

VECTOR ALGEBRA:

VECTOR ADDITION:

If the vectors $\vec{A}$ and $\vec{B}$ are expressed as,

$$\vec{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}$$

$$\vec{B} = B_x \vec{a_x} + B_y \vec{a_y} + B_z \vec{a_z}$$

The Addition of vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A} + \vec{B} = (A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}) + (B_x \vec{a_x} + B_y \vec{a_y} + B_z \vec{a_z})$$

$$\boxed{\vec{A} + \vec{B} = (A_x + B_x)\vec{a_x} + (A_y + B_y)\vec{a_y} + (A_z + B_z)\vec{a_z}}$$

VECTOR SUBTRACTION:

The Subtraction of vectors $\vec{A}$ and $\vec{B}$ is

$$\vec{A} - \vec{B} = (A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}) - (B_x \vec{a_x} + B_y \vec{a_y} + B_z \vec{a_z})$$

$$\boxed{\vec{A} - \vec{B} = (A_x - B_x)\vec{a_x} + (A_y - B_y)\vec{a_y} + (A_z - B_z)\vec{a_z}}$$

PROBLEM:

① If $\vec{A} = 3\vec{a_x} + 2\vec{a_y} + 6\vec{a_z}$ and $\vec{B} = 2\vec{a_x} + 5\vec{a_y} - 3\vec{a_z}$. find

(i) $\vec{A} + \vec{B}$

(ii) $\vec{A} - \vec{B}$

Given:

$$\vec{A} = 3\vec{a_x} + 2\vec{a_y} + 6\vec{a_z}$$

$$\vec{B} = 2\vec{a_x} + 5\vec{a_y} - 3\vec{a_z}$$

$$\vec{A} + \vec{B} = 3\vec{a_x} + 2\vec{a_y} + 6\vec{a_z} + 2\vec{a_x} + 5\vec{a_y} - 3\vec{a_z}$$

$$\boxed{\vec{A} + \vec{B} = 5\vec{a_x} + 7\vec{a_y} + 3\vec{a_z}}$$

$$\vec{A} - \vec{B} = 3\vec{a_x} + 2\vec{a_y} + 6\vec{a_z} - 2\vec{a_x} - 5\vec{a_y} + 3\vec{a_z}$$

$$\boxed{\vec{A} - \vec{B} = \vec{a_x} - 3\vec{a_y} + 9\vec{a_z}}$$

VECTOR MULTIPLICATION:

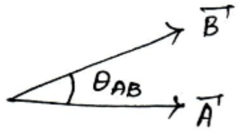
There are two types of vector multiplication

(i) Dot Product (or) Scalar product

(ii) Cross Product (or) Vector Product.

DOT PRODUCT:

Dot Product or Scalar product is defined as the product of magnitude of two vectors and Cosine of the angle between them.



$$\boxed{\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos \theta_{AB}}$$

PROPERTIES OF DOT PRODUCT:

(i) If two vectors $\vec{A}$ and $\vec{B}$ are parallel to each other $(\theta_{AB} = 0^\circ)$

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|$$

(ii) If two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other $(\theta_{AB} = 90^\circ)$

$$\vec{A} \cdot \vec{B} = 0$$

(iii) The dot product obeys Commutative law.

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

(iv) The dot product obeys distributive law

$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

(v) If $\vec{a_x}$, $\vec{a_y}$ and $\vec{a_z}$ are unit vectors of cartesian system

$$\vec{a_x} \cdot \vec{a_x} = \vec{a_y} \cdot \vec{a_y} = \vec{a_z} \cdot \vec{a_z} = 1$$

$$\vec{a_x} \cdot \vec{a_y} = \vec{a_y} \cdot \vec{a_z} = \vec{a_z} \cdot \vec{a_x} = 0$$

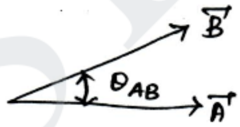
(vi) If $\vec{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}$ and $\vec{B} = B_x \vec{a_x} + B_y \vec{a_y} + B_z \vec{a_z}$

$$\vec{A} \cdot \vec{B} = (A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}) \cdot (B_x \vec{a_x} + B_y \vec{a_y} + B_z \vec{a_z})$$

$$= A_x B_x + A_y B_y + A_z B_z.$$

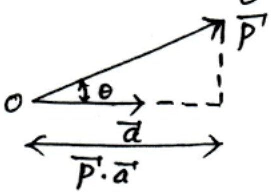
APPLICATIONS OF THE DOT PRODUCT:

(i) To determine the angle between two vectors.



$$\theta_{AB} = \cos^{-1}\left[\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}\right]$$

(ii) To find the Component of a vector in a given direction (i,e) Projection of a vector in given direction.



$$\left.\begin{array}{l}\text{Component of } \vec{P} \text{ in the direction of } \vec{a} \\ \text{(or)} \\ \text{Projection of } \vec{P} \text{ in the direction of } \vec{a}\end{array}\right\} (\vec{P} \cdot \vec{a}) = |\vec{P}| \cos \theta.$$

(iii) To find the workdone by a constant force.

$$W = |\vec{F}|\, d \cos\theta = \vec{F} \cdot \vec{d}$$

PROBLEM:

① Given the two vectors $\vec{A} = 3\vec{a_x} - \vec{a_y} + 8\vec{a_z}$ and $\vec{B} = 2\vec{a_x} + 3\vec{a_y} - \vec{a_z}$. Find the dot product and angle between the two vectors.

Given:

$$\vec{A} = 3\vec{a_x} - \vec{a_y} + 8\vec{a_z}$$

$$\vec{B} = 2\vec{a_x} + 3\vec{a_y} - \vec{a_z}$$

Solution:

$$\vec{A} \cdot \vec{B} = (3\vec{a_x} - \vec{a_y} + 8\vec{a_z}) \cdot (2\vec{a_x} + 3\vec{a_y} - \vec{a_z})$$

$$= (3)(2) + (-1)(3) + (8)(-1)$$

$$= 6 - 3 - 8$$

$$\boxed{\vec{A} \cdot \vec{B} = -5}$$

$$|\vec{A}| = \sqrt{(3)^2 + (-1)^2 + (8)^2} = \sqrt{9+1+64} = \sqrt{74}$$

$$|\vec{B}| = \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{4+9+1} = \sqrt{14}$$

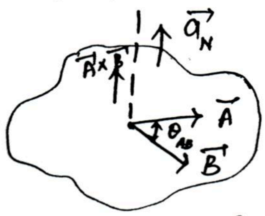
$$\theta = \cos^{-1}\left[\frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}\right] = \cos^{-1}\left[\frac{-5}{\sqrt{74}\sqrt{14}}\right]$$

$$\boxed{\theta = 98.93^\circ}$$

CROSS PRODUCT:

The Cross Product or Vector product is defined as the product of magnitude of two vectors and sine of the angle between them.

The Cross product of two vectors is a vector. So it can be expressed with unit normal vector perpendicular to the plane of two vectors.



$$\boxed{\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta_{AB}\, \vec{a_N}}$$

PROPERTIES OF CROSS PRODUCT:

(i) If two vectors $\vec{A}$ and $\vec{B}$ are parallel to each other $(\theta_{AB} = 0^\circ)$

$$\vec{A} \times \vec{B} = 0.$$

(ii) If two vectors $\vec{A}$ and $\vec{B}$ are perpendicular to each other ($\because \theta_{AB} = 90^\circ$)

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\,\vec{a}_N$$

(iii) The Cross product doesn't obey Commutative law.

$$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$$

$$\vec{A} \times B = -(\vec{B} \times \vec{A})$$

(iv) The Cross product obeys distributive law.

$$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$$

(v) If $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ are the unit vectors of Cartesian System

$$\vec{a}_x \times \vec{a}_y = \vec{a}_z, \quad \vec{a}_y \times \vec{a}_z = \vec{a}_x, \quad \vec{a}_z \times \vec{a}_x = \vec{a}_y$$

$$\vec{a}_y \times \vec{a}_x = -\vec{a}_z, \quad \vec{a}_z \times \vec{a}_y = -\vec{a}_x, \quad \vec{a}_x \times \vec{a}_z = -\vec{a}_y$$

$$\vec{a}_x \times \vec{a}_x = \vec{a}_y \times \vec{a}_y = \vec{a}_z \times \vec{a}_z = 0.$$

(vi) If $\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$ and $\vec{B} = B_x \vec{a}_x + B_y \vec{a}_y + B_z \vec{a}_z$

$$\vec{A} \times \vec{B} = (A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z) \times (B_x \vec{a}_x + B_y \vec{a}_y + B_z \vec{a}_z)$$

$$= \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

APPLICATIONS OF CROSS PRODUCT:

(i) The Cross product is the replacement of right handed rule in Electrical Engineering.

(ii) To find the moment of force.

PROBLEM:

① Given the two vectors $\vec{A} = \vec{a}_x - 5\vec{a}_y + 2\vec{a}_z$ and $\vec{B} = 3\vec{a}_x - \vec{a}_y - 4\vec{a}_z$. find the cross product and the unit normal vector.

Given:

$$\vec{A} = \vec{a}_x - 5\vec{a}_y + 2\vec{a}_z$$

$$\vec{B} = 3\vec{a}_x - \vec{a}_y - 4\vec{a}_z$$

Solution:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ 1 & -5 & 2 \\ 3 & -1 & -4 \end{vmatrix}$$

$$= \vec{a_x}(20+2) - \vec{a_y}(-4-6) + \vec{a_z}(-1+15)$$

$$\boxed{\vec{A} \times \vec{B} = 22\vec{a_x} + 10\vec{a_y} + 14\vec{a_z}}$$

$$|\vec{A} \times \vec{B}| = \sqrt{484+100+196} = \sqrt{780}$$

$$\Rightarrow \quad \vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \, \vec{a_N} \quad \text{——— (1)}$$

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta \quad \text{——— (2)} \quad \because |\vec{a_N}| = 1$$

$$\frac{(1)}{(2)} \Rightarrow \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \vec{a_N}$$

$$\vec{a_N} = \frac{22\vec{a_x} + 10\vec{a_y} + 14\vec{a_z}}{\sqrt{780}}$$

$$\boxed{\vec{a_N} = \frac{1}{\sqrt{780}}(22\vec{a_x} + 10\vec{a_y} + 14\vec{a_z})}$$

Multiplication of Three Vectors:

Dot Product of Three Vectors / Scalar Triple Product:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

If $\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}$, $\vec{B} = B_x\vec{a_x} + B_y\vec{a_y} + B_z\vec{a_z}$ and $\vec{C} = C_x\vec{a_x} + C_y\vec{a_y} + C_z\vec{a_z}$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Properties of Scalar Triple Product:

(i) Scalar triple product represents the volume of parallelepiped.

(ii) $\vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{B} \cdot (\vec{A} \times \vec{C})$

(iii) If two of three vectors are equal, then the result of the scalar triple product is zero.

$$\vec{A} \cdot (\vec{A} \times \vec{C}) = 0$$

(iv) The scalar triple product is distributive.

PROBLEM:

① Given the three vectors $\vec{A} = 2\vec{a_x} - \vec{a_z}$, $\vec{B} = 2\vec{a_x} - \vec{a_y} + 2\vec{a_z}$ and $\vec{C} = 2\vec{a_x} - 3\vec{a_y} + \vec{a_z}$. Find the scalar triple product.

Given: $\vec{A} = 2\vec{a_x} - \vec{a_z}$

$\vec{B} = 2\vec{a_x} - \vec{a_y} + 2\vec{a_z}$

$\vec{C} = 2\vec{a_x} - 3\vec{a_y} + \vec{a_z}$

Solution:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} 2 & 0 & -1 \\ 2 & -1 & 2 \\ 2 & -3 & 1 \end{vmatrix}$$

$$= 2(-1+6) - 0(2-4) - 1(-6+2)$$

$$= 2(4) - 1(-4)$$

$$= 8 + 4$$

$$\boxed{\vec{A}\cdot(\vec{B}\times\vec{C}) = 12}$$

CROSS PRODUCT OF THREE VECTORS / VECTOR TRIPLE PRODUCT:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

[Hint: BAC- CAB Rule]

PROPERTIES OF VECTOR TRIPLE PRODUCT:

(i) $(\vec{A}\cdot\vec{B})\vec{C} \neq \vec{A}(\vec{B}\cdot\vec{C})$

$(\vec{A}\cdot\vec{B})\vec{C} = \vec{C}(\vec{A}\cdot\vec{B})$

(ii) $\vec{B}\times(\vec{C}\times\vec{A}) = \vec{C}(\vec{B}\cdot\vec{A}) - \vec{A}(\vec{B}\cdot\vec{C})$

$\vec{C}\times(\vec{A}\times\vec{B}) = \vec{A}(\vec{C}\cdot\vec{B}) - \vec{B}(\vec{C}\cdot\vec{A})$

(iii) $\vec{A}\times(\vec{B}\times\vec{C}) + \vec{B}\times(\vec{C}\times\vec{A}) + \vec{C}\times(\vec{A}\times\vec{B}) = 0$

PROBLEM:

① Given the three vectors $\vec{A} = 2\vec{a_x} - \vec{a_z}$, $\vec{B} = 2\vec{a_x} - \vec{a_y} + 2\vec{a_z}$ and $\vec{C} = 2\vec{a_x} - 3\vec{a_y} + \vec{a_z}$. Find the vector triple product.

Given: $\vec{A} = 2\vec{a_x} - \vec{a_z}$

$\vec{B} = 2\vec{a_x} - \vec{a_y} + 2\vec{a_z}$

$\vec{C} = 2\vec{a_x} - 3\vec{a_y} + \vec{a_z}$

Solution:

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\vec{A}\cdot\vec{C} = (2\vec{a}_x - \vec{a}_z)\cdot(2\vec{a}_x - 3\vec{a}_y + \vec{a}_z)$$

$$= 4 - 0 - 1$$

$$= 3$$

$$\vec{A}\cdot\vec{B} = (2\vec{a}_x - \vec{a}_z)\cdot(2\vec{a}_x - \vec{a}_y + 2\vec{a}_z)$$

$$= 4 - 2$$

$$= 2$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = 3\vec{B} - 2\vec{C}$$

$$= 3(2\vec{a}_x - \vec{a}_y + 2\vec{a}_z) - 2(2\vec{a}_x - 3\vec{a}_y + \vec{a}_z)$$

$$= 6\vec{a}_x - 3\vec{a}_y + 6\vec{a}_z - 4\vec{a}_x + 6\vec{a}_y - 2\vec{a}_z$$

$$\boxed{\vec{A}\times(\vec{B}\times\vec{C}) = 2\vec{a}_x + 3\vec{a}_y + 4\vec{a}_z}$$

COORDINATE SYSTEMS:

NEED OF COORDINATE SYSTEMS:

- To describe the spatial variations of the quantities.
- To define all points uniquely in space in suitable manner.

Coordinate Systems

Orthogonal Coordinate Systems

- The Coordinates of the system are mutually perpendicular.

Non Orthogonal Coordinate Systems.

- The Coordinates of the System are not mutually perpendicular.

Examples:

(i) Cartesian or Rectangular

(ii) Cylindrical or Circular

(iii) Spherical

(iv) Elliptic Cylindrical

(v) Parabolic Cylindrical

(vi) Conical

(vii) Prolate Spheroidal

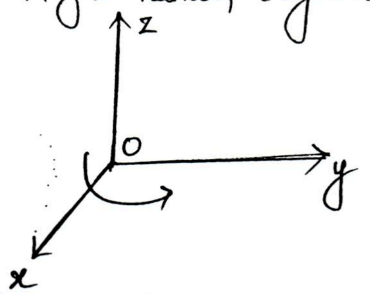
(viii) Oblate Spheroidal

(ix) Ellipsoidal.

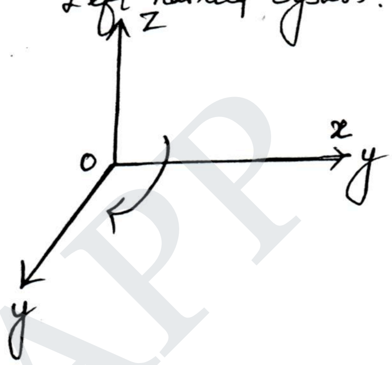
CARTESIAN COORDINATE SYSTEM / RECTANGULAR COORDINATE SYSTEM :

- The Coordinates of a Rectangular Coordinate System are x, y and z.
- The Rectangular Coordinate System has three coordinate axes represented as x, y and z which are mutually right angles to each other.
- These three axes intersects at a Common point Called origin.
- There are two types,

Right handed System



Left handed System.



RANGE OF VARIABLES :

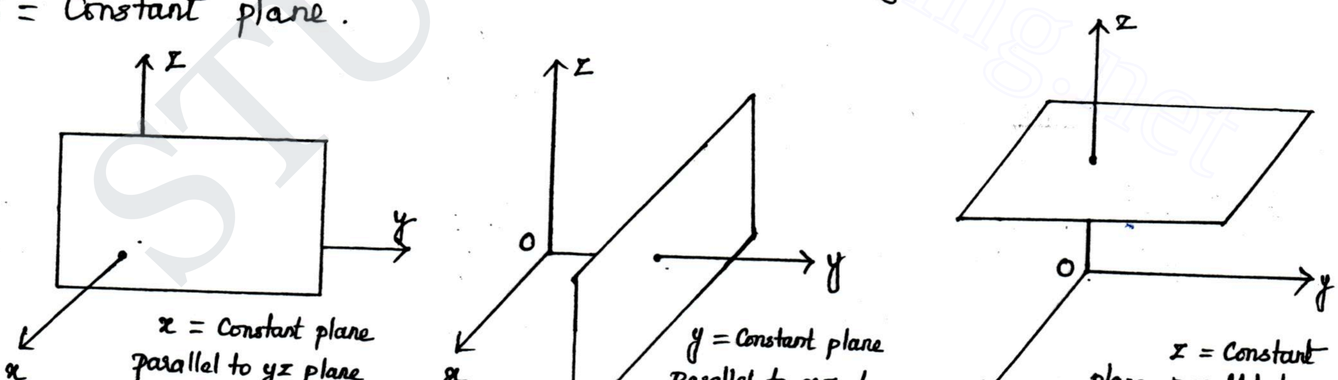
$$-\infty \leq x \leq \infty$$

$$-\infty \leq y \leq \infty$$

$$-\infty \leq z \leq \infty$$

REPRESENTATION OF A POINT :

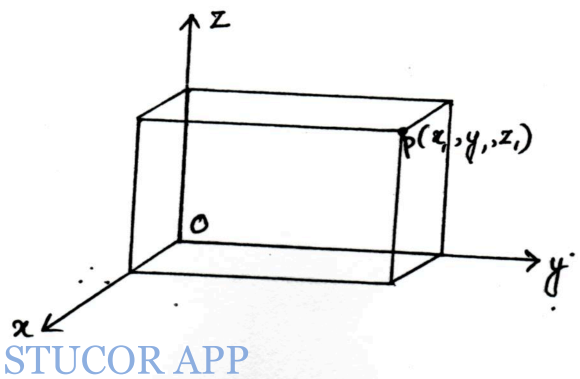
A point in Rectangular Coordinate System can be represented as the intersection of x = constant plane, y = Constant plane and z = Constant plane.



x = Constant plane parallel to yz plane

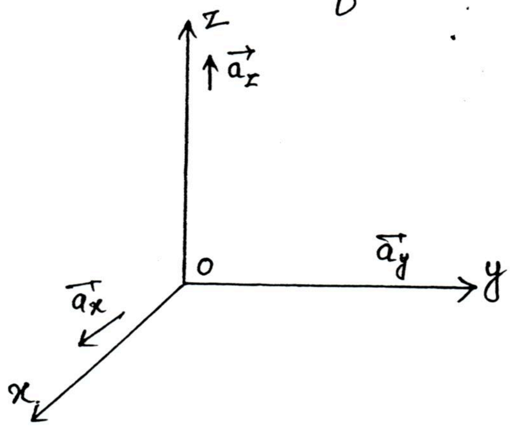
y = Constant plane Parallel to xz plane

z = Constant plane parallel to xy plane.

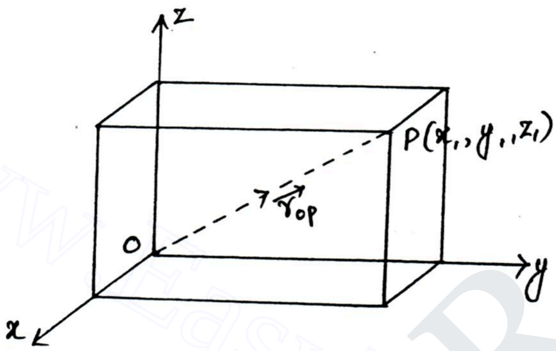


Base Vectors :

The base vectors are the unit vectors which are strictly oriented along the directions of the Coordinate Systems.

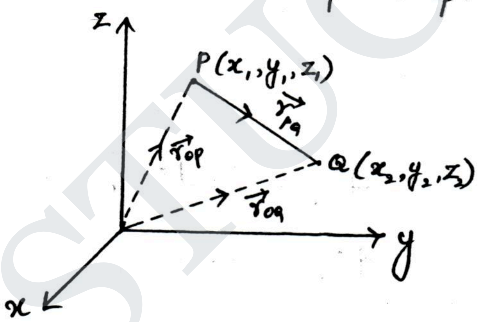


Position and Distance Vectors :



Position vector/ Radius vector $\Big\}$ $\vec{r}_{op} = x_1 \vec{a_x} + y_1 \vec{a_y} + z_1 \vec{a_z}$

$$r_{op} = |\vec{r}_{op}| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$



$$\vec{r}_{op} = x_1 \vec{a_x} + y_1 \vec{a_y} + z_1 \vec{a_z}$$

$$\vec{r}_{oQ} = x_2 \vec{a_x} + y_2 \vec{a_y} + z_2 \vec{a_z}$$

Distance vector/ Separation vector $\Big\}$ $\vec{r}_{PQ} = \vec{r}_{oQ} - \vec{r}_{op}$

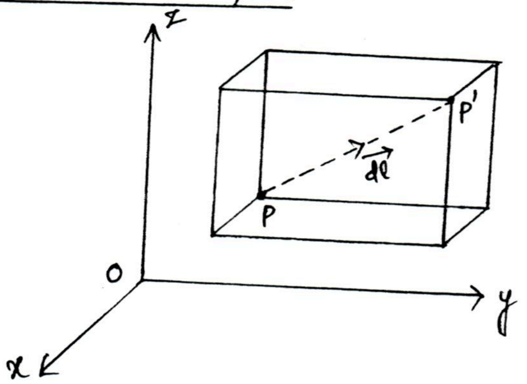
$$= (x_2 - x_1)\vec{a_x} + (y_2 - y_1)\vec{a_y} + (z_2 - z_1)\vec{a_z}$$

Distance, $r_{PQ} = |\vec{r}_{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Unit vector along PQ $\Big\}$ $\vec{a}_{PQ} = \dfrac{\vec{r}_{PQ}}{|\vec{r}_{PQ}|} = \dfrac{(x_2 - x_1)\vec{a_x} + (y_2 - y_1)\vec{a_y} + (z_2 - z_1)\vec{a_z}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$

Differential Elements:

(a) Differential Length:



$P \to (x, y, z)$

$P' \to (x+dx, y+dy, z+dz)$

- Let a point $P(x, y, z)$ in the rectangular Coordinate System.
- By increasing each coordinate by differential amount a new point $P'(x+dx, y+dy, z+dz)$ obtained.

dx – Differential length in x direction

dy – Differential length in y direction

dz – Differential length in z direction

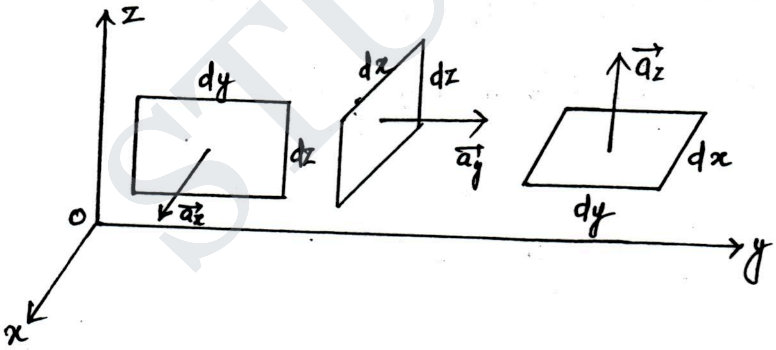
Differential vector length / Elemental vector length } $\vec{dl} = dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}$

$$dl = |\vec{dl}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

(b) Differential Surface:

$$\vec{ds} = ds\,\vec{a_n}$$

where $\vec{a_n}$ – Unit vector normal to the surface ds.



$\vec{dS_x} = dy\,dz\,\vec{a_x}$

$\vec{dS_y} = dx\,dz\,\vec{a_y}$

$\vec{dS_z} = dx\,dy\,\vec{a_z}$

(c) Differential Volume:

Differential Volume } $dV = dx\,dy\,dz$

Representation of a Vector:

The Vector in Rectangular Coordinate System can be expressed as

$$\vec{A} = A_x\,\vec{a_x} + A_y\,\vec{a_y} + A_z\,\vec{a_z}.$$

Circular Coordinate System / Cylindrical Coordinate System :

– Circular Coordinate system is a three dimensional version of polar coordinate system.

– The Coordinates of a Cylindrical Coordinate System are ρ, ϕ and z.

– Surfaces used to define a cylindrical coordinate system are,....

(a) Constant z plane which is parallel to xy plane.

(b) A Cylinder of radius ρ with z axis as the axis of the Cylinder.

(c) A half plane perpendicular to xy plane at an angle of ϕ with respect to xz plane.

Range of Variables :

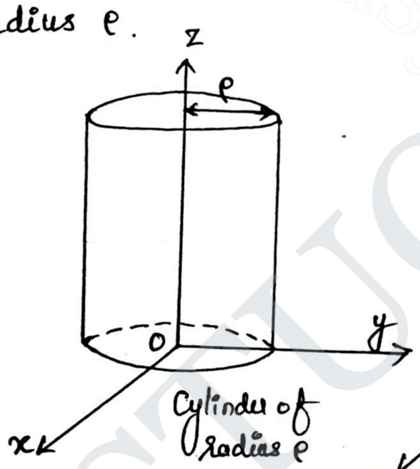
$$0 \le \rho \le \infty$$

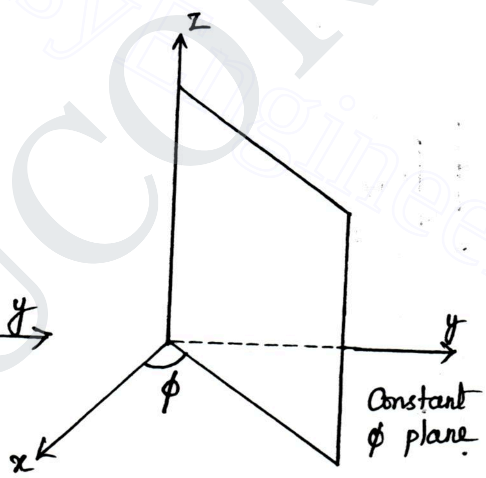
$$0 \le \phi \le 2\pi$$

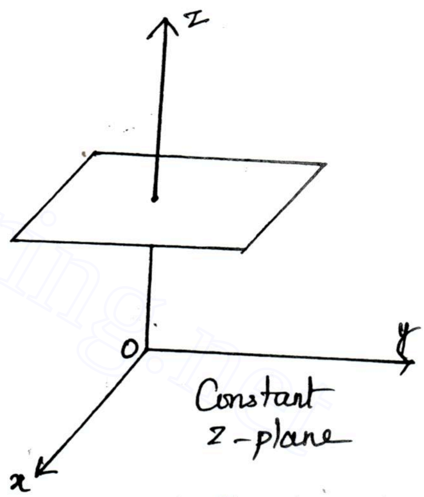
$$-\infty \le z \le \infty$$

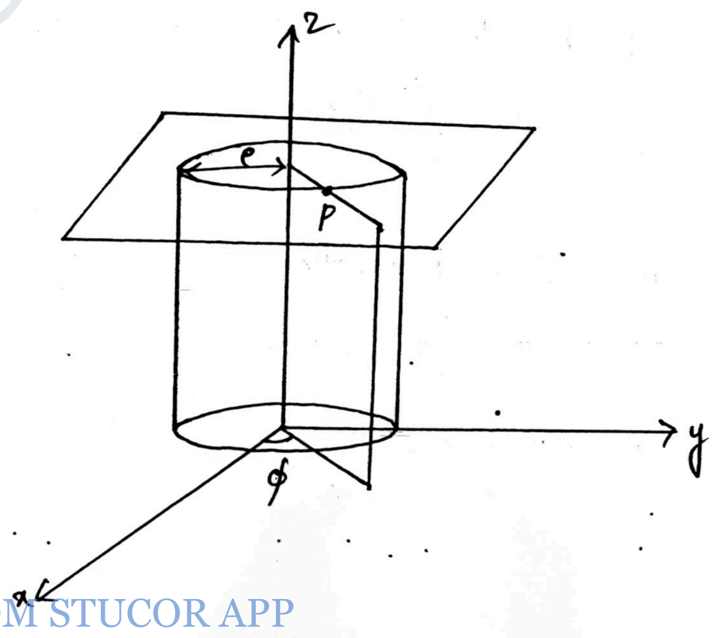
Representation of a Point :

A point in Cylindrical Coordinate System can be represented as the intersection of Constant z plane, Constant ϕ plane and a cylinder of radius ρ.



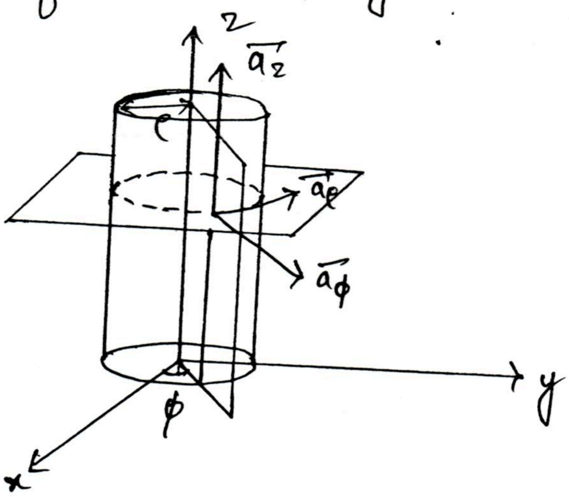






Base Vectors:

The base vectors are the unit vectors which are strictly oriented along the directions of the coordinate systems.



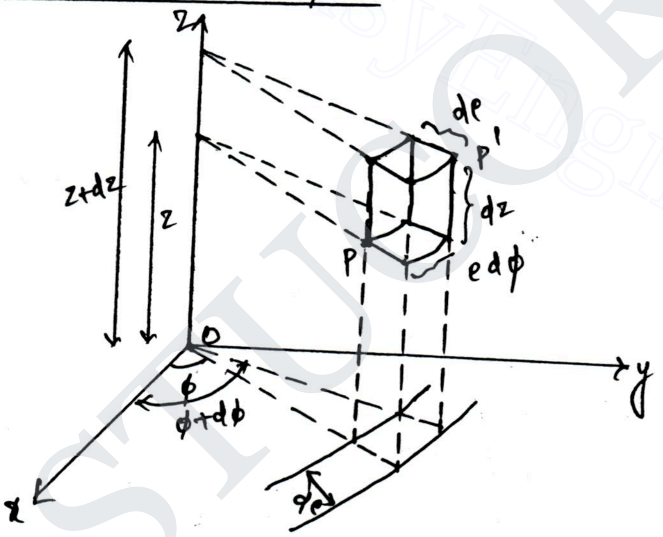
Distance:

The distance between the two points in cylindrical coordinate system is,

$$d = \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}.$$

Differential Elements:

(a) Differential Length:



- Let a point $P(\rho, \phi, z)$ in the cylindrical coordinate system.
- By increasing each coordinate by differential amount a new point P' obtained.

$d\rho$ – Differential length in ρ direction

$\rho d\phi$ – Differential length in ϕ direction

dz – Differential length in z direction.

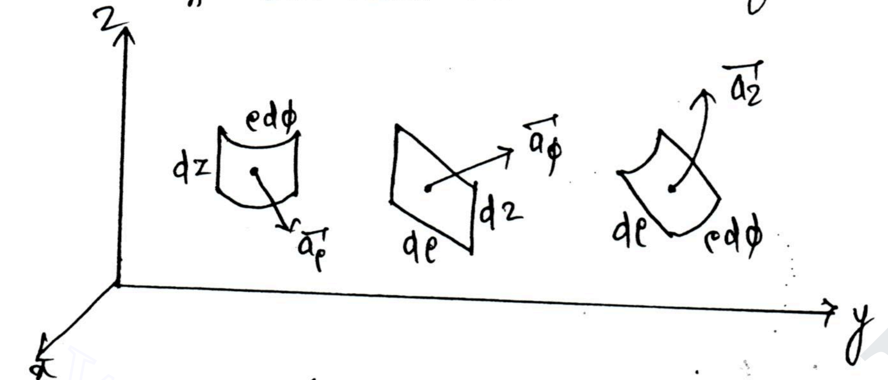
Differential Vector length, $\vec{dl} = d\rho\,\vec{a_\rho} + \rho\,d\phi\,\vec{a_\phi} + dz\,\vec{a_z}$

$$dl = |\vec{dl}| = \sqrt{(d\rho)^2 + (\rho\,d\phi)^2 + (dz)^2}$$

(b) DIFFERENTIAL SURFACE:

$$\vec{ds} = ds\,\vec{a_n}$$

where $\vec{a_n}$ – Unit Vector normal to the Surface ds.



$$\vec{ds_\rho} = \rho\,d\phi\,dz\,\vec{a_\rho}$$

$$\vec{ds_\phi} = d\rho\,dz\,\vec{a_\phi}$$

$$\vec{ds_z} = \rho\,d\rho\,d\phi\,\vec{a_z}$$

(c) DIFFERENTIAL VOLUME:

Differential Volume, $dV = \rho\,d\rho\,d\phi\,dz$.

REPRESENTATION OF A VECTOR:

The vector in Cylindrical Coordinate System can be expressed as

$$\vec{A} = A_\rho\,\vec{a_\rho} + A_\phi\,\vec{a_\phi} + A_z\,\vec{a_z}.$$

SPHERICAL COORDINATE SYSTEM:

- The Coordinates of a Spherical Coordinate system are r, θ and ϕ.
- Surfaces used to define a Spherical Coordinate System are,

(a) A Sphere of radius r, origin as the Center of the Sphere

(b) A right circular Cone with its apex at the origin and its axis as z-axis. Its half angle is θ. It rotates about z axis and θ varies from 0 to 180°.

(c) A half plane perpendicular to xy plane Containing z-axis, making an angle ϕ with the xz plane.

RANGE OF VARIABLES:

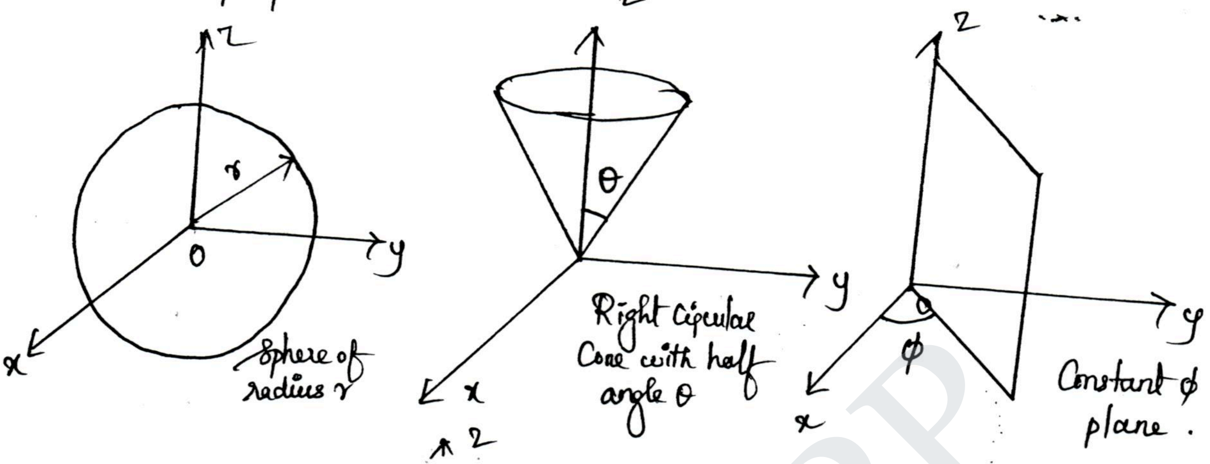
$$0 \le r \le \infty$$

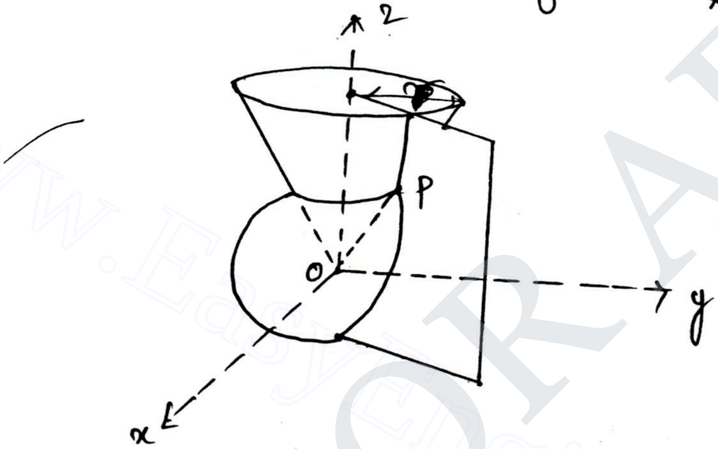
$$0 \le \phi \le 2\pi$$

$$0 \le \theta \le \pi$$

Representation of a Point:

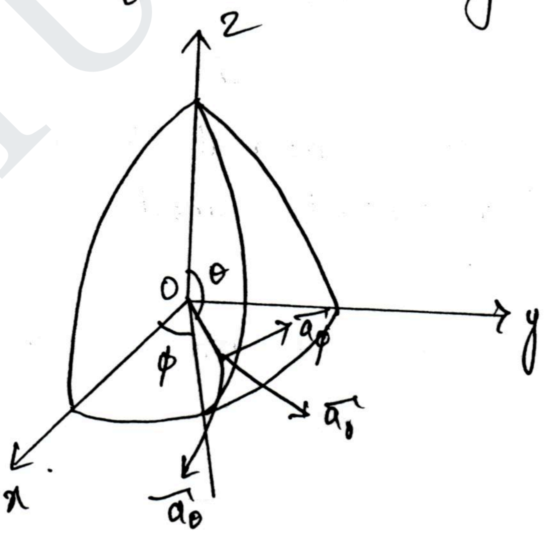
A point in Spherical Coordinate System can be represented as the intersection of sphere of radius r, right circular cone with half angle θ and constant ϕ plane.





Base Vectors:

The base vectors are the unit vectors which are strictly oriented along the directions of the coordinate systems.



Distance:

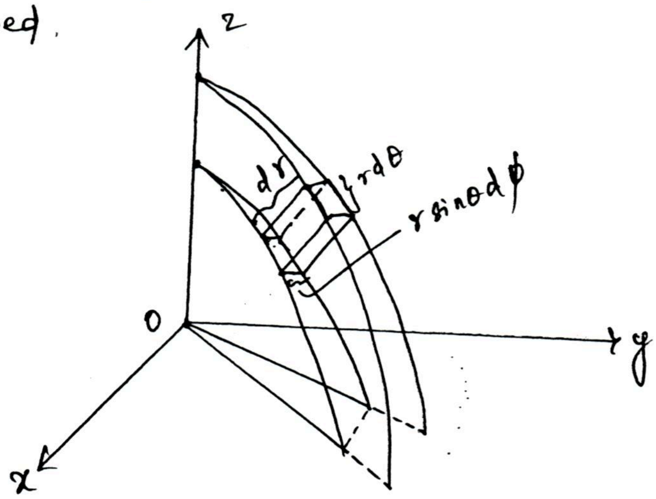
The distance between the two points in Spherical Coordinate System is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta_1\cos\theta_2 - 2r_1r_2\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1)}.$$

DIFFERENTIAL ELEMENTS:

(a) DIFFERENTIAL LENGTH:

- Let a point $P(r, \theta, \phi)$ in Spherical Coordinate System.
- By increasing each coordinate by differential amount a new point P' obtained.



dr – Differential length in r direction

$rd\theta$ – Differential length in θ direction

$r\sin\theta\, d\phi$ – Differential length in ϕ direction.

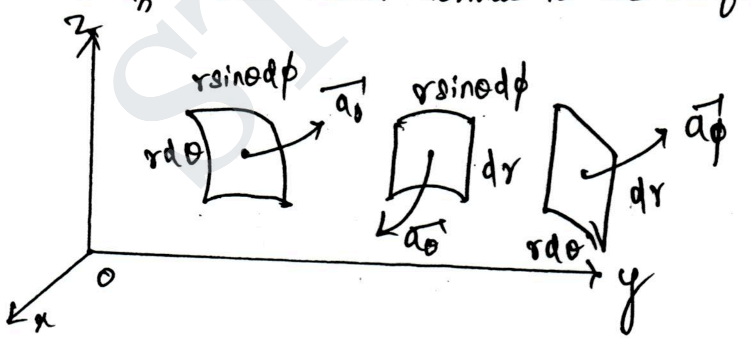
Differential Vector Length, $\vec{dl} = dr\,\vec{a_r} + rd\theta\,\vec{a_\theta} + r\sin\theta\, d\phi\,\vec{a_\phi}$

$$dl = |\vec{dl}| = \sqrt{(dr)^2 + (rd\theta)^2 + (r\sin\theta\, d\phi)^2}$$

(b) DIFFERENTIAL SURFACE:

$$\vec{ds} = ds\,\vec{a_n}$$

where $\vec{a_n}$ – Unit Vector normal to the Surface ds.



$$\vec{ds_r} = r^2\sin\theta\, d\theta\, d\phi\,\vec{a_r}$$

$$\vec{ds_\theta} = r\sin\theta\, dr\, d\phi\,\vec{a_\theta}$$

$$\vec{ds_\phi} = r\,dr\,d\theta\,\vec{a_\phi}$$

(c) DIFFERENTIAL VOLUME:

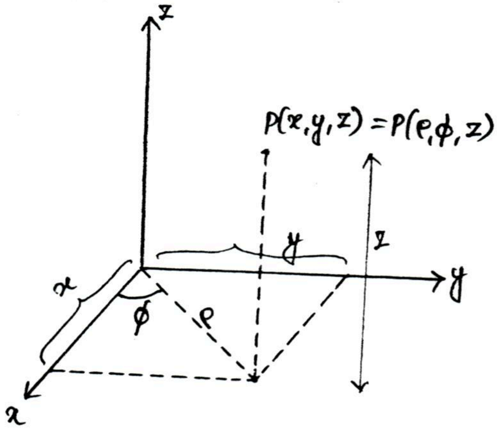
Differential Volume, $dV = r^2\sin\theta\, dr\, d\theta\, d\phi$

REPRESENTATION OF A VECTOR:

A vector in Spherical Coordinate System can be expressed as

$$\vec{A} = A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi}.$$

RELATION BETWEEN CARTESIAN AND CYLINDRICAL COORDINATE SYSTEMS:



$$\cos\phi = \frac{x}{\rho}$$

$$\boxed{x = \rho\cos\phi}$$

$$\sin\phi = \frac{y}{\rho}$$

$$\boxed{y = \rho\sin\phi}$$

$$x^2+y^2 = \rho^2\cos^2\phi + \rho^2\sin^2\phi$$

$$= \rho^2(\cos^2\phi + \sin^2\phi)$$

$$x^2+y^2 = \rho^2$$

$$\boxed{\rho = \sqrt{x^2+y^2}}$$

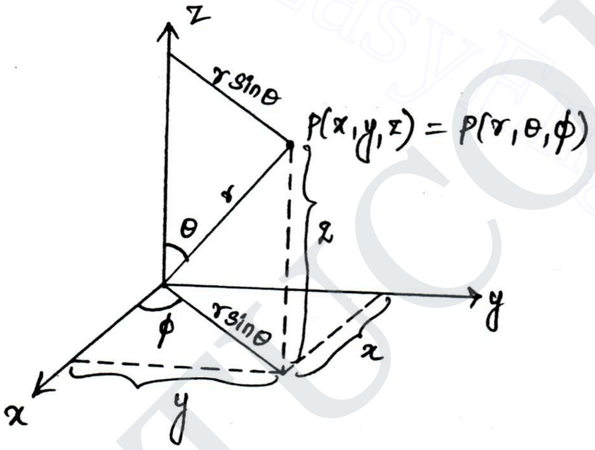
$$\frac{y}{x} = \frac{\rho\sin\phi}{\rho\cos\phi}$$

$$\frac{y}{x} = \tan\phi$$

$$\boxed{\phi = \tan^{-1}\left(\frac{y}{x}\right)}$$

$$\boxed{z = z}$$

RELATION BETWEEN CARTESIAN AND SPHERICAL COORDINATE SYSTEMS:



$$\cos\phi = \frac{x}{r\sin\theta}$$

$$\boxed{x = r\sin\theta\cos\phi}$$

$$\sin\phi = \frac{y}{r\sin\theta}$$

$$\boxed{y = r\sin\theta\sin\phi}$$

$$\cos\theta = \frac{z}{r}$$

$$\boxed{z = r\cos\theta}$$

$$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$

$$= r^2\sin^2\theta(\cos^2\phi+\sin^2\phi) + r^2\cos^2\theta$$

$$= r^2\sin^2\theta + r^2\cos^2\theta$$

$$= r^2(\sin^2\theta + \cos^2\theta)$$

$$= r^2$$

$$\boxed{r = \sqrt{x^2+y^2+z^2}}$$

$$\frac{y}{x} = \frac{\cancel{r}\cancel{\sin\theta}\sin\phi}{\cancel{r}\cancel{\sin\theta}\cos\phi}$$

$$\frac{y}{x} = \tan\phi$$

$$\boxed{\phi = \tan^{-1}\left(\frac{y}{x}\right)}$$

$$z = r\cos\theta$$

$$\cos\theta = \frac{z}{r}$$

$$\boxed{\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}}$$

S.No	Coordinate Systems	Coordinate Variable	Range of Variables	Unit Vectors	Differential Length (dl)	Differential Area (ds)	Differential Volume (dV)
1	Cartesian/ Rectangular	x y z	$-\infty \le x \le \infty$ $-\infty \le y \le \infty$ $-\infty \le z \le \infty$	$\vec{a_x}$ $\vec{a_y}$ $\vec{a_z}$	$\vec{dl} = dx\vec{a_x} + dy\vec{a_y} + dz\vec{a_z}$ $dl = \sqrt{(dx)^2+(dy)^2+(dz)^2}$	$\vec{ds_x} = dy\,dz\,\vec{a_x}$ $\vec{ds_y} = dx\,dz\,\vec{a_y}$ $\vec{ds_z} = dx\,dy\,\vec{a_z}$	$dV = dx\,dy\,dz$
2	Circular/ Cylindrical	ρ ϕ z	$0 \le \rho \le \infty$ $0 \le \phi \le 2\pi$ $-\infty \le z \le \infty$	$\vec{a_\rho}$ $\vec{a_\phi}$ $\vec{a_z}$	$\vec{dl} = d\rho\vec{a_\rho} + \rho d\phi\vec{a_\phi} + dz\vec{a_z}$ $dl = \sqrt{(d\rho)^2+(\rho d\phi)^2+(dz)^2}$	$\vec{ds_\rho} = \rho\,d\phi\,dz\,\vec{a_\rho}$ $\vec{ds_\phi} = d\rho\,dz\,\vec{a_\phi}$ $\vec{ds_z} = \rho\,d\rho\,d\phi\,\vec{a_z}$	$dV = \rho\,d\rho\,d\phi\,dz$
3	Spherical	r θ ϕ	$0 \le r \le \infty$ $0 \le \theta \le \pi$ $0 \le \phi \le 2\pi$	$\vec{a_r}$ $\vec{a_\theta}$ $\vec{a_\phi}$	$\vec{dl} = dr\vec{a_r} + r d\theta\vec{a_\theta} + r\sin\theta d\phi\vec{a_\phi}$ $dl = \sqrt{(dr)^2+(rd\theta)^2+(r\sin\theta d\phi)^2}$	$\vec{ds_r} = r^2\sin\theta\,d\theta\,d\phi\,\vec{a_r}$ $\vec{ds_\theta} = r\sin\theta\,dr\,d\phi\,\vec{a_\theta}$ $\vec{ds_\phi} = r\,dr\,d\theta\,\vec{a_\theta}$	$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

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PROBLEMS:

① Give the Cylindrical Coordinates of the point whose Cartesian Coordinates are $x = 3$, $y = 4$ and $z = 5$ units.

Given: $x = 3$, $y = 4$, $z = 5$

Solution:

$$\rho = \sqrt{x^2+y^2} = \sqrt{9+16} = \sqrt{25}$$

$$\boxed{\rho = 5}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\boxed{\phi = 53.13^\circ}$$

$$z = z$$

$$\boxed{z = 5}$$

The cylindrical Coordinates are $\rho = 5$, $\phi = 53.13^\circ$, $z = 5$

② Give the Cartesian Coordinates of the point whose Cylindrical Coordinates are $\rho = 2$, $\phi = 45^\circ$ and $z = -1$

Given: $\rho = 2$, $\phi = 45^\circ$, $z = -1$

Solution:

$$x = \rho\cos\phi = 2\cos 45^\circ$$

$$\boxed{x = 0.707}$$

$$y = \rho\sin\phi = 2\sin 45^\circ$$

$$\boxed{y = 0.707}$$

$$z = z$$

$$\boxed{z = -1}$$

The Cartesian Coordinates are $x = 0.707$, $y = 0.707$, $z = -1$.

③ Give the Spherical Coordinates of the point whose Cartesian Coordinates are $x = -1$, $y = 3$ and $z = 5$

Given: $x = -1$, $y = 3$, $z = 5$

Solution:

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1+9+25} = \sqrt{35}$$

$$\boxed{r = \quad}$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{-1}\right) = \tan^{-1}(-3)$$

$$\boxed{\phi = \quad}$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \cos^{-1}\left(\frac{5}{\quad}\right)$$

$$\boxed{\theta = \quad}$$

The Spherical Coordinates are $r =$, $\theta =$, $\phi =$

(4) Give the Cartesian Coordinates of the point whose Spherical Coordinates are $r = 3$, $\theta = 60°$ and $\phi = 30°$

Given: $r = 3$, $\theta = 60°$, $\phi = 30°$

Solution:

$x = r \sin\theta \cos\phi$

$= 3 \sin 60° \cos 30°$

$= 3 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$

$= \frac{9}{4}$

$\boxed{x = 2.25}$

$y = r \sin\theta \sin\phi$

$= 3 \sin 60° \sin 30°$

$= 3 \times \frac{\sqrt{3}}{2} \times \frac{1}{2}$

$\boxed{y = }$

$z = r \cos\theta$

$= 3 \cos 60°$

$= \frac{3}{2}$

$\boxed{z = 1.5}$

The Cartesian Coordinates are $x = 2.25$, $y =$, $z = 1.5$.

(5) Given Points $A(x = 2, y = 3, z = -1)$ and $B(\rho = 4, \phi = -50°, z = 2)$. Find the distance from A to B.

Given: $A(x = 2, y = 3, z = -1)$

$B(\rho = 4, \phi = -50°, z = 2)$

$x = \rho \cos\phi$

$= 4 \cos(-50°)$

$\boxed{x = 2.571}$

$y = \rho \sin\phi$

$= 4 \sin(-50°)$

$\boxed{y = -3.064}$

$z = z$

$\boxed{z = 2}$

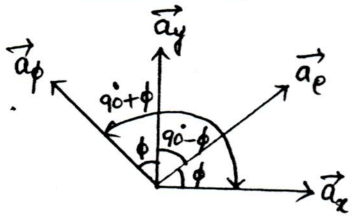
$B(\rho = 4, \phi = -50°, z = 2) = B(x = 2.571, y = -3.064, z = 2)$

$$\left.\begin{array}{l}\text{Distance from} \\ \text{A to B}\end{array}\right\} = \sqrt{(2.571 - 2)^2 + (-3.064 - 3)^2 + (2+1)^2}$$

$$= \sqrt{46.098}$$

$$\boxed{d = 6.789}$$

DOT PRODUCT BETWEEN THE UNIT VECTORS OF CARTESIAN AND CYLINDRICAL COORDINATE SYSTEMS:



- $\vec{a}_x$ & $\vec{a}_y$ are perpendicular (i.e) $\vec{a}_x \cdot \vec{a}_y = 0$
- $\vec{a}_\rho$ & $\vec{a}_\phi$ are perpendicular (i.e) $\vec{a}_\rho \cdot \vec{a}_\phi = 0$
- $\vec{a}_z$ is perpendicular to $\vec{a}_x$ & $\vec{a}_y$ (i.e) $\vec{a}_z \cdot \vec{a}_x = \vec{a}_z \cdot \vec{a}_y = 0$
- $\vec{a}_z$ is perpendicular to $\vec{a}_\rho$ & $\vec{a}_\phi$ (i.e) $\vec{a}_z \cdot \vec{a}_\rho = \vec{a}_z \cdot \vec{a}_\phi = 0$

$$\vec{a}_x \cdot \vec{a}_\rho = |\vec{a}_x||\vec{a}_\rho| \cos\phi = (1)(1)\cos\phi = \cos\phi$$

$$\vec{a}_x \cdot \vec{a}_\phi = (1)(1)\cos(90^\circ+\phi) = -\sin\phi$$

$$\vec{a}_x \cdot \vec{a}_z = 0$$

$$\vec{a}_y \cdot \vec{a}_\rho = (1)(1)\cos(90^\circ-\phi) = \sin\phi$$

$$\vec{a}_y \cdot \vec{a}_\phi = (1)(1)\cos\phi = \cos\phi$$

$$\vec{a}_y \cdot \vec{a}_z = 0$$

$$\vec{a}_z \cdot \vec{a}_\rho = 0$$

$$\vec{a}_z \cdot \vec{a}_\phi = 0$$

$$\vec{a}_z \cdot \vec{a}_z = 1$$

$\cdot$	$\vec{a}_\rho$	$\vec{a}_\phi$	$\vec{a}_z$
$\vec{a}_x$	$\cos\phi$	$-\sin\phi$	0
$\vec{a}_y$	$\sin\phi$	$\cos\phi$	0
$\vec{a}_z$	0	0	1

DOT PRODUCT BETWEEN THE UNIT VECTORS OF CATESIAN AND SPHERICAL COORDINATE SYSTEMS:

$\cdot$	$\vec{a}_r$	$\vec{a}_\theta$	$\vec{a}_\phi$
$\vec{a}_x$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\vec{a}_y$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\vec{a}_z$	$\cos\theta$	$-\sin\theta$	0

The component of A in the direction of $\vec{a_x}$ is the dot product of $\vec{A}$ & $\vec{a_x}$.

$$A_x = \vec{A}\cdot\vec{a_x} = (A_\rho\vec{a_\rho}+A_\phi\vec{a_\phi}+A_z\vec{a_z})\cdot\vec{a_x} = A_\rho(\vec{a_\rho}\cdot\vec{a_x})+A_\phi(\vec{a_\phi}\cdot\vec{a_x})+A_z(\vec{a_z}\cdot\vec{a_x})$$

$$A_x = A_\rho\cos\phi - A_\phi\sin\phi$$

$$A_y = \vec{A}\cdot\vec{a_y} = (A_\rho\vec{a_\rho}+A_\phi\vec{a_\phi}+A_z\vec{a_z})\cdot\vec{a_y} = A_\rho(\vec{a_\rho}\cdot\vec{a_y})+A_\phi(\vec{a_\phi}\cdot\vec{a_y})+A_z(\vec{a_z}\cdot\vec{a_y})$$

$$A_y = A_\rho\sin\phi + A_\phi\cos\phi$$

$$A_z = \vec{A}\cdot\vec{a_z} = (A_\rho\vec{a_\rho}+A_\phi\vec{a_\phi}+A_z\vec{a_z})\cdot\vec{a_z} = A_\rho(\vec{a_\rho}\cdot\vec{a_z})+A_\phi(\vec{a_\phi}\cdot\vec{a_z})+A_z(\vec{a_z}\cdot\vec{a_z})$$

$$A_z = A_z$$

The result of the transformation in matrix form is

$$\boxed{\begin{bmatrix}A_x\\A_y\\A_z\end{bmatrix} = \begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}A_\rho\\A_\phi\\A_z\end{bmatrix}}$$

TRANSFORMATION OF VECTORS FROM CARTESIAN TO SPHERICAL COORDINATE SYSTEMS:

Consider a vector in Cartesian coordinate system as

$$\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}$$

The same vector $\vec{A}$ in spherical coordinate system can be expressed as

$$\vec{A} = A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi}$$

The component of $\vec{A}$ in the direction of $\vec{a_r}$ is the dot product of $\vec{A}$ & $\vec{a_r}$

$$A_r = \vec{A}\cdot\vec{a_r} = (A_x\vec{a_x}+A_y\vec{a_y}+A_z\vec{a_z})\cdot\vec{a_r} = A_x(\vec{a_x}\cdot\vec{a_r})+A_y(\vec{a_y}\cdot\vec{a_r})+A_z(\vec{a_z}\cdot\vec{a_r})$$

$$A_r = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$$

$$A_\theta = \vec{A}\cdot\vec{a_\theta} = (A_x\vec{a_x}+A_y\vec{a_y}+A_z\vec{a_z})\cdot\vec{a_\theta} = A_x(\vec{a_x}\cdot\vec{a_\theta})+A_y(\vec{a_y}\cdot\vec{a_\theta})+A_z(\vec{a_z}\cdot\vec{a_\theta})$$

$$A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$$

$$A_\phi = \vec{A}\cdot\vec{a_\phi} = (A_x\vec{a_x}+A_y\vec{a_y}+A_z\vec{a_z})\cdot\vec{a_\phi} = A_x(\vec{a_x}\cdot\vec{a_\phi})+A_y(\vec{a_y}\cdot\vec{a_\phi})+A_z(\vec{a_z}\cdot\vec{a_\phi})$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$

The result of the transformation in matrix form is,

$$\boxed{\begin{bmatrix}A_r\\A_\theta\\A_\phi\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}A_x\\A_y\\A_z\end{bmatrix}}$$

Dot Product Between the Unit Vectors of Cylindrical and Spherical Coordinate Systems:

$\cdot$	$\vec{a}_r$	$\vec{a}_\theta$	$\vec{a}_\phi$
$\vec{a}_e$	$\sin\theta$	$\cos\theta$	0
$\vec{a}_\phi$	0	0	1
$\vec{a}_z$	$\cos\theta$	$-\sin\theta$	0

Transformation of Vectors from Cartesian to Cylindrical Coordinate Systems:

Consider a vector $\vec{A}$ in Cartesian Coordinate system as

$$\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

The same vector $\vec{A}$ in Cylindrical Coordinate system can be expressed as

$$\vec{A} = A_e\vec{a}_e + A_\phi\vec{a}_\phi + A_z\vec{a}_z$$

The Component of $\vec{A}$ in the direction of $\vec{a}_e$ is the dot product of $\vec{A}$ & $\vec{a}_e$

$$A_e = \vec{A}\cdot\vec{a}_e = (A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z)\cdot\vec{a}_e = A_x(\vec{a}_x\cdot\vec{a}_e) + A_y(\vec{a}_y\cdot\vec{a}_e) + A_z(\vec{a}_z\cdot\vec{a}_e)$$

$$A_e = A_x\cos\phi + A_y\sin\phi$$

$$A_\phi = \vec{A}\cdot\vec{a}_\phi = (A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z)\cdot\vec{a}_\phi = A_x(\vec{a}_x\cdot\vec{a}_\phi) + A_y(\vec{a}_y\cdot\vec{a}_\phi) + A_z(\vec{a}_z\cdot\vec{a}_\phi)$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$

$$A_z = \vec{A}\cdot\vec{a}_z = (A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z)\cdot\vec{a}_z = A_x(\vec{a}_x\cdot\vec{a}_z) + A_y(\vec{a}_y\cdot\vec{a}_z) + A_z(\vec{a}_z\cdot\vec{a}_z)$$

$$A_z = A_z$$

The Result of the transformation in matrix form is

$$\boxed{\begin{bmatrix} A_e \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}}$$

Transformation of Vectors from Cylindrical to Cartesian Coordinate Systems:

Consider a vector $\vec{A}$ in Cylindrical Coordinate system as

$$\vec{A} = A_e\vec{a}_e + A_\phi\vec{a}_\phi + A_z\vec{a}_z$$

The same vector in Cartesian Coordinate system can be expressed as

$$\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

 The result of the transformation in matrix form is

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

TRANSFORMATION OF VECTORS FROM SPHERICAL TO CYLINDRICAL COORDINATE SYSTEMS:

Consider a vector $\vec{A}$ in spherical coordinate system as

$$\vec{A} = A_r \vec{a_r} + A_\theta \vec{a_\theta} + A_\phi \vec{a_\phi}$$

The same vector $\vec{A}$ in cylindrical coordinate system can be expressed as

$$\vec{A} = A_\rho \vec{a_\rho} + A_\phi \vec{a_\phi} + A_z \vec{a_z}$$

The component of $\vec{A}$ in the direction of $\vec{a_\rho}$ is the dot product of $\vec{A}$ and $\vec{a_\rho}$.

$$A_\rho = \vec{A}\cdot\vec{a_\rho} = (A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi})\cdot\vec{a_\rho} = A_r(\vec{a_r}\cdot\vec{a_\rho}) + A_\theta(\vec{a_\theta}\cdot\vec{a_\rho}) + A_\phi(\vec{a_\phi}\cdot\vec{a_\rho})$$

$$A_\rho = A_r \sin\theta + A_\theta \cos\theta$$

$$A_\phi = \vec{A}\cdot\vec{a_\phi} = (A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi})\cdot\vec{a_\phi} = A_r(\vec{a_r}\cdot\vec{a_\phi}) + A_\theta(\vec{a_\theta}\cdot\vec{a_\phi}) + A_\phi(\vec{a_\phi}\cdot\vec{a_\phi})$$

$$A_\phi = A_\phi$$

$$A_z = \vec{A}\cdot\vec{a_z} = (A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi})\cdot\vec{a_z} = A_r(\vec{a_r}\cdot\vec{a_z}) + A_\theta(\vec{a_\theta}\cdot\vec{a_z}) + A_\phi(\vec{a_\phi}\cdot\vec{a_z})$$

$$A_z = A_r \cos\theta - A_\theta \sin\theta$$

The result of the transformation in matrix form is,

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Consider a vector $\vec{A}$ in Spherical Coordinate System as

$$\vec{A} = A_r \vec{a_r} + A_\theta \vec{a_\theta} + A_\phi \vec{a_\phi}$$

The same vector $\vec{A}$ in Cartesian Coordinate System can be expressed as.

$$\vec{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}$$

The Component of $\vec{A}$ in the direction of $\vec{a_x}$ is the dot product of $\vec{A}$ & $\vec{a_x}$

$$A_x = \vec{A}\cdot\vec{a_x} = (A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi})\cdot\vec{a_x} = A_r(\vec{a_r}\cdot\vec{a_x}) + A_\theta(\vec{a_\theta}\cdot\vec{a_x}) + A_\phi(\vec{a_\phi}\cdot\vec{a_x})$$

$$A_x = A_r \sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$$

$$A_y = \vec{A}\cdot\vec{a_y} = (A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi})\cdot\vec{a_y} = A_r(\vec{a_r}\cdot\vec{a_y}) + A_\theta(\vec{a_\theta}\cdot\vec{a_y}) + A_\phi(\vec{a_\phi}\cdot\vec{a_y})$$

$$A_y = A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$$

$$A_z = \vec{A}\cdot\vec{a_z} = (A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi})\cdot\vec{a_z} = A_r(\vec{a_r}\cdot\vec{a_z}) + A_\theta(\vec{a_\theta}\cdot\vec{a_z}) + A_\phi(\vec{a_\phi}\cdot\vec{a_z})$$

$$A_z = A_r\cos\theta - A_\theta\sin\theta$$

The result of the transformation in matrix form is

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}}$$

TRANSFORMATION OF VECTORS FROM CYLINDRICAL TO SPHERICAL COORDINATE SYSTEM:

Consider a vector $\vec{A}$ in Cylindrical Coordinate System as

$$\vec{A} = A_\rho\vec{a_\rho} + A_\phi\vec{a_\phi} + A_z\vec{a_z}$$

The same vector $\vec{A}$ in Spherical Coordinate system can be expressed as

$$\vec{A} = A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi}$$

The Component of $\vec{A}$ in the direction of $\vec{a_r}$ is the dot product of $\vec{A}$ and $\vec{a_r}$.

$$A_r = \vec{A}\cdot\vec{a_r} = (A_\rho\vec{a_\rho} + A_\phi\vec{a_\phi} + A_z\vec{a_z})\cdot\vec{a_r} = A_\rho(\vec{a_\rho}\cdot\vec{a_r}) + A_\phi(\vec{a_\phi}\cdot\vec{a_r}) + A_z(\vec{a_z}\cdot\vec{a_r})$$

$$A_r = A_\rho\sin\theta + A_z\cos\theta$$

$$A_\theta = \vec{A}\cdot\vec{a_\theta} = (A_\rho\vec{a_\rho} + A_\phi\vec{a_\phi} + A_z\vec{a_z})\cdot\vec{a_\theta} = A_\rho(\vec{a_\rho}\cdot\vec{a_\theta}) + A_\phi(\vec{a_\phi}\cdot\vec{a_\theta}) + A_z(\vec{a_z}\cdot\vec{a_\theta})$$

$$A_\theta = A_\rho\cos\theta - A_z\sin\theta$$

$$A_\phi = \vec{A}\cdot\vec{a_\phi} = (A_\rho\vec{a_\rho} + A_\phi\vec{a_\phi} + A_z\vec{a_z})\cdot\vec{a_\phi} = A_\rho(\vec{a_\rho}\cdot\vec{a_\phi}) + A_\phi(\vec{a_\phi}\cdot\vec{a_\phi}) + A_z(\vec{a_z}\cdot\vec{a_\phi})$$

$$A_\phi = A_\phi$$

PROBLEM:

① The scalar fields are given by

(i) $\alpha = 20e^{-x} \sin\left(\frac{\pi}{6}\right) y$

(ii) $\alpha = 25\rho \sin\phi$

(iii) $\alpha = \frac{40 \cos\theta}{r^2}$

Find its gradient at the point $P(0,1,1)$ for Cartesian, $P(\sqrt{2}, \frac{\pi}{2}, 5)$ for Cylindrical, $P(3, 60^\circ, 30^\circ)$ for Spherical.

Given:

$\alpha = 20e^{-x} \sin\left(\frac{\pi}{6}\right) y$ at $P(0,1,1)$

$\alpha = 25\rho \sin\phi$ at $P(\sqrt{2}, \frac{\pi}{2}, 5)$

$\alpha = \frac{40 \cos\theta}{r^2}$ at $P(3, 60^\circ, 30^\circ)$

Solution:

For Cartesian:

$$\nabla\alpha = \frac{\partial\alpha}{\partial x}\vec{a_x} + \frac{\partial\alpha}{\partial y}\vec{a_y} + \frac{\partial\alpha}{\partial z}\vec{a_z}$$

$$\frac{\partial\alpha}{\partial x} = -20e^{-x} \sin\left(\frac{\pi}{6}\right) y$$

$$\frac{\partial\alpha}{\partial y} = 20e^{-x} \frac{\pi}{6} \cos\left(\frac{\pi}{6}\right) y$$

$$\frac{\partial\alpha}{\partial z} = 0$$

$$\nabla\alpha = \left[-20e^{-x} \sin\left(\frac{\pi}{6}\right) y\right]\vec{a_x} + \left[20e^{-x} \frac{\pi}{6} \cos\left(\frac{\pi}{6}\right) y\right]\vec{a_y}$$

$$(\nabla\alpha)_{(0,1,1)} = \left[-20 \sin\frac{\pi}{6}\right]\vec{a_x} + \left[20 \times \frac{\pi}{6} \times \cos\frac{\pi}{6}\right]\vec{a_y}$$

$$\boxed{(\nabla\alpha)_{(0,1,1)} = -10\vec{a_x} + 9.07\vec{a_y}}$$

VECTOR DIFFERENTIAL OPERATOR (∇):

The Vector differential operator ∇ Can be written as

$$\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$$

There are three operations involved with the vector differential Operator

(i) Gradient

(ii) Divergence

(iii) Curl.

GRADIENT:

Gradient of any scalar function is the maximum rate of change of the function.

Let V be the scalar potential function. The gradient of V in Cartesian Coordinate System as

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z$$

in Cylindrical Coordinate System as

$$\nabla V = \frac{\partial V}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z$$

in Spherical Coordinate System as

$$\nabla V = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$$

PROPERTIES OF GRADIENT:

(i) The gradient ∇V gives the maximum rate of change of V per unit distance.

(ii) The gradient ∇V always indicates the direction of the maximum rate of change of V.

(iii) The gradient ∇V at any point is perpendicular to the Constant V surface, which passes through the point.

(iv) If U and V are Scalars

$$\nabla(U+V) = \nabla U + \nabla V$$

$$\nabla(UV) = U(\nabla V) + V(\nabla U)$$

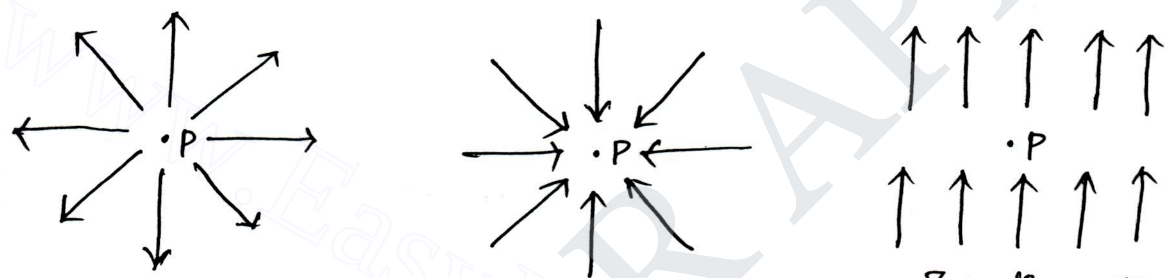
$$\nabla\left(\frac{U}{V}\right) = \frac{V(\nabla U) - U(\nabla V)}{V^2}$$

DIVERGENCE:

The divergence of a vector field $\vec{A}$ at a given point P is the outward flux per unit volume as the volume shrinks about P.

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \lim_{\Delta V \to 0} \frac{\oint_S \vec{A} \cdot d\vec{s}}{\Delta V}$$

- It is a measure of how much the field diverges or emanates from that point.
- Divergence is +Ve at a source point in the field
- Divergence is -Ve at a sink point in the field
- Divergence is zero there is neither sink nor source.



Positive divergence   Negative divergence   Zero divergence

— If $\vec{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}$

for Cartesian Coordinate System

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

for Cylindrical Coordinate System

$$\nabla \cdot \vec{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

for Spherical Coordinate System

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}.$$

PROPERTIES OF DIVERGENCE:

(i) It produces a scalar field

(ii) $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

(iii) $\nabla \cdot (V\vec{A}) = V(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla V)$

(iv) If $\nabla \cdot \vec{A} = 0 \Rightarrow$ The vector field is said to be Solenoidal field.

for cylindrical:

$$\nabla\alpha = \frac{\partial \alpha}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial \alpha}{\partial \phi}\vec{a}_\phi + \frac{\partial \alpha}{\partial z}\vec{a}_z$$

$$\frac{\partial \alpha}{\partial \rho} = 25\sin\phi$$

$$\frac{\partial \alpha}{\partial \phi} = 25\rho\cos\phi$$

$$\frac{\partial \alpha}{\partial z} = 0$$

$$\nabla\alpha = [25\sin\phi]\vec{a}_\rho + \frac{1}{\cancel{\rho}}[25\cancel{\rho}\cos\phi]\vec{a}_\phi$$

$$(\nabla\alpha)_{(\sqrt{2},\frac{\pi}{2},5)} = \left[25\sin\frac{\pi}{2}\right]\vec{a}_\rho + \left[25\cos\frac{\pi}{2}\right]\vec{a}_\phi$$

$$\boxed{(\nabla\alpha)_{(\sqrt{2},\frac{\pi}{2},5)} = 25\vec{a}_\rho}$$

for Spherical:

$$\nabla\alpha = \frac{\partial \alpha}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial \alpha}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial \alpha}{\partial \phi}\vec{a}_\phi$$

$$\frac{\partial \alpha}{\partial r} = \frac{-80\cos\theta}{r^3}$$

$$\frac{\partial \alpha}{\partial \theta} = \frac{-40\sin\theta}{r^2}$$

$$\frac{\partial \alpha}{\partial \phi} = 0$$

$$\nabla\alpha = \left[\frac{-80\cos\theta}{r^3}\right]\vec{a}_r + \frac{1}{r}\left[\frac{-40\sin\theta}{r^2}\right]\vec{a}_\theta$$

$$(\nabla\alpha)_{(3,60^\circ,30^\circ)} = \left[\frac{-80\cos 60^\circ}{27}\right]\vec{a}_r - \left[\frac{40\sin 60^\circ}{27}\right]\vec{a}_\theta$$

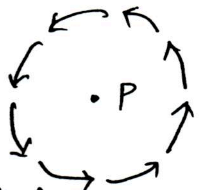
$$\boxed{(\nabla\alpha)_{(3,60^\circ,30^\circ)} = -1.48\vec{a}_r - 1.28\vec{a}_\theta}$$

CURL:

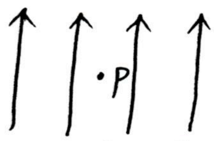
The Curl of $\vec{A}$ is an axial or rotational vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the surface tends to zero. And whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \left(\lim_{\Delta s \to 0} \frac{\oint \vec{A} \cdot d\vec{L}}{\Delta S} \right)_{max} \vec{a}_n$$

where $\vec{a}_n$ – Unit vector normal to the surface ΔS.



Curl at P points out of the page.



Curl at P is zero

For Cartesian Coordinate System

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \vec{A} = \vec{a}_x \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right] + \vec{a}_y \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right] + \vec{a}_z \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]$$

For Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \vec{a}_\rho & \rho \vec{a}_\phi & \vec{a}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla \times \vec{A} = \left[\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] \vec{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right] \vec{a}_\phi + \frac{1}{\rho} \left[\frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right] \vec{a}_z$$

for Spherical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\, \vec{a}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta} \left[\frac{\partial (A_\phi \sin\theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right] \vec{a}_r + \frac{1}{r} \left[\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r} \right] \vec{a}_\theta + \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right] \vec{a}_\phi$$

① Determine the divergence of the vector fields.

(i) $\vec{P} = x^2yz\,\vec{a}_x + xz\,\vec{a}_z$

(ii) $\vec{Q} = \rho\sin\phi\,\vec{a}_\rho + \rho^2 z\,\vec{a}_\phi + z\cos\phi\,\vec{a}_z$.

(iii) $\vec{T} = \frac{1}{r^2}\cos\theta\,\vec{a}_r + r\sin\theta\cos\phi\,\vec{a}_\theta + \cos\theta\,\vec{a}_\phi$

Solution:

<u>for Cartesian:</u>

$$\nabla\cdot\vec{P} = \frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}$$

<u>Given:</u> $P_x = x^2yz$, $P_y = 0$, $P_z = xz$

$$\nabla\cdot\vec{P} = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$

$$\boxed{\nabla\cdot\vec{P} = 2xyz + x}$$

<u>for Cylindrical:</u>

$$\nabla\cdot\vec{Q} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial Q_\phi}{\partial\phi} + \frac{\partial Q_z}{\partial z}$$

<u>Given:</u> $Q_\rho = \rho\sin\phi$, $Q_\phi = \rho^2 z$, $Q_z = z\cos\phi$

$$\nabla\cdot\vec{Q} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$

$$= \frac{1}{\rho}(2\rho\sin\phi) + \frac{1}{\rho}(0) + \cos\phi$$

$$\boxed{\nabla\cdot\vec{Q} = 2\sin\phi + \cos\phi}$$

<u>for Spherical:</u>

$$\nabla\cdot\vec{T} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial T_\phi}{\partial\phi}$$

Given: $T_r = \frac{1}{r^2}\cos\theta$, $T_\theta = r\sin\theta\cos\phi$, $T_\phi = \cos\theta$

$$\nabla\cdot\vec{T} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\times\frac{1}{\cancel{r^2}}\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta)$$

$$= \frac{1}{r^2}(0) + \frac{\cancel{r}\cos\phi}{\cancel{r}\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1-\cos 2\theta}{2}\right) + \frac{1}{r\sin\theta}(0)$$

$$= \frac{\cos\phi}{\sin\theta}\left(\frac{\cancel{2}\sin 2\theta}{\cancel{2}}\right) = \frac{\cos\phi}{\cancel{\sin\theta}}\times 2\cancel{\sin\theta}\cos\theta$$

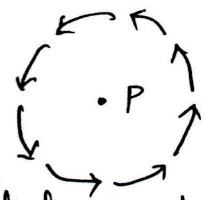
$$\boxed{\nabla\cdot\vec{T} = 2\cos\theta\cos\phi}$$

CURL :

The Curl of $\vec{A}$ is an axial or rotational vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the surface tends to zero. And whose direction is the normal direction of the area when the area is oriented to make the circulation maximum.

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \left(\lim_{\Delta S \to 0} \frac{\oint \vec{A} \cdot \vec{dL}}{\Delta S} \right)_{max} \vec{a_n}$$

where $\vec{a_n}$ – Unit vector normal to the surface ΔS.



Curl at P points out of the page.

Curl at P is zero

For Cartesian Coordinate System

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \vec{A} = \vec{a_x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \vec{a_y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \vec{a_z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

For Cylindrical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \vec{a_\rho} & \rho\vec{a_\phi} & \vec{a_z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\nabla \times \vec{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\vec{a_\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\vec{a_\phi} + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right]\vec{a_z}$$

For Spherical Coordinate System

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \vec{a_r} & r\vec{a_\theta} & r\sin\theta\,\vec{a_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta\,A_\phi \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi \sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]\vec{a_r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\vec{a_\theta} + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\vec{a_\phi}$$

For Spherical:

$$\nabla\times\vec{T} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a_r} & r\vec{a_\theta} & r\sin\theta\,\vec{a_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ T_r & rT_\theta & r\sin\theta T_\phi \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a_r} & r\vec{a_\theta} & r\sin\theta\,\vec{a_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ \frac{1}{r^2}\cos\theta & r\sin\theta\cos\phi & \cos\theta \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\vec{a_r}\left(-\sin\theta + r\sin\theta\sin\phi\right) - r\vec{a_\theta}(0-0) + r\sin\theta\,\vec{a_\phi}\left(\sin\theta\cos\phi + \frac{1}{r^2}\sin\theta\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\sin\theta\,(r\sin\phi - 1)\,\vec{a_r} + r\sin^2\theta\left(\cos\phi + \frac{1}{r^2}\right)\vec{a_\phi}\right]$$

$$\boxed{\nabla\times\vec{T} = \left(\frac{\sin\phi}{r} - \frac{1}{r^2}\right)\vec{a_r} + \frac{\sin\theta}{r}\left(\cos\phi + \frac{1}{r^2}\right)\vec{a_\phi}}$$

② Prove that $\nabla\cdot(\nabla\times\vec{A}) = 0$

(or)

Prove that divergence of curl of $\vec{A}$ is zero.

Solution:

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \vec{a_x}\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right] + \vec{a_y}\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right] + \vec{a_z}\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]$$

$$\nabla\cdot(\nabla\times\vec{A}) = \left[\frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}\right]\cdot\left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{a_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{a_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{a_z}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x\partial y} - \frac{\partial^2 A_y}{\partial x\partial z} + \frac{\partial^2 A_x}{\partial y\partial z} - \frac{\partial^2 A_z}{\partial x\partial y} + \frac{\partial^2 A_y}{\partial x\partial z} - \frac{\partial^2 A_x}{\partial y\partial z}$$

$$\boxed{\nabla\cdot(\nabla\times\vec{A}) = 0}$$

 Consider the elemental volume in x - direction.

$$So, A_y = A_z = 0$$

$$\therefore \nabla.\vec{A} = \frac{\partial A_x}{\partial x}$$

$$\iiint \nabla.\vec{A}\, dv = \iiint \frac{\partial A_x}{\partial x}\, dx\, dy\, dz \qquad \because dv = dx\,dy\,dz$$

$$= \iint A_x\, dy\, dz$$

$$= \iint A_x\, dS_x \quad \text{——— (1)} \qquad \because dS_x = dy\,dz$$

Consider the elemental volume in y - direction

$$So, A_x = A_z = 0$$

$$\therefore \nabla.\vec{A} = \frac{\partial A_y}{\partial y}$$

$$\iiint \nabla.\vec{A}\, dv = \iiint \frac{\partial A_y}{\partial y}\, dx\, dy\, dz$$

$$= \iint A_y\, dx\, dz$$

$$= \iint A_y\, dS_y \quad \text{——— (2)} \qquad \because dS_y = dx\,dz$$

Consider the elemental volume in z - direction

$$So\ A_x = A_y = 0$$

$$\therefore \nabla.\vec{A} = \frac{\partial A_z}{\partial z}$$

$$\iiint \nabla.\vec{A}\, dv = \iiint \frac{\partial A_z}{\partial z}\, dx\, dy\, dz$$

$$= \iint A_z\, dx\, dy$$

$$= \iint A_z\, dS_z \quad \text{——— (3)} \qquad \because dS_z = dx\,dy$$

from equation (1), (2) and (3)

$$\iiint_V \nabla.\vec{A}\, dv = \iint A_x\, dS_x + \iint A_y\, dS_y + \iint A_z\, dS_z$$

$$= \iint (A_x ds_x + A_y ds_y + A_z ds_z)$$

$$= \iint (A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}) \cdot (ds_x \vec{a_x} + ds_y \vec{a_y} + ds_z \vec{a_z})$$

$$\boxed{\iiint_V (\nabla \cdot \vec{A}) dv = \iint_S \vec{A} \cdot \vec{ds}}$$

Hence Proved.

PROBLEM:

① The Vector field $\vec{D} = \frac{5r^2}{4} \vec{a_r}$ is given in Spherical Coordinate system. Evaluate both sides of divergence theorem for the volume is enclosed between

(i) $r = 1$ and $r = 2$

(ii) $\theta = 0$ to $\frac{\pi}{4}$ and $r = 4$

Solution:

Case (i): $r = 1$ & $r = 2$.

By divergence theorem

$$\iiint_V (\nabla \cdot \vec{D}) dv = \iint_S \vec{D} \cdot \vec{ds}$$

LHS: $\iiint_V (\nabla \cdot \vec{D}) dv$

$$\nabla \cdot \vec{D} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \times \frac{5r^2}{4} \right) = \frac{1}{r^2} \times \frac{5}{4} \times 4r^3$$

$$\nabla \cdot \vec{D} = 5r$$

$$\iiint \nabla \cdot \vec{D} dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=1}^{2} 5r \times r^2 \sin\theta \, dr \, d\theta \, d\phi \qquad \because dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 5 \int_0^{2\pi} \int_0^{\pi} \int_1^2 r^3 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 5\int_0^{2\pi}\int_0^{\pi}\left[\frac{r^4}{4}\right]_1^2 \sin\theta\, d\theta\, d\phi$$

$$= 5\left(\frac{16-1}{4}\right)\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi} d\phi$$

$$= \frac{75}{4}\int_0^{2\pi}(1+1)\, d\phi$$

$$= \frac{75}{2}\left[\phi\right]_0^{2\pi}$$

$$= \frac{75}{2}\times 2\pi$$

$$\iiint \nabla\cdot\vec{D}\, dV = 75\pi \quad \text{—————} (1)$$

RHS: $\oint \vec{D}\cdot\vec{ds}$

$$\iint \vec{D}\cdot\vec{ds} = \iint_{r=2} \vec{D}\cdot ds_r\, \vec{a_r} - \iint_{r=1} \vec{D}\cdot ds_r\, \vec{a_r}$$

$$= \iint_{r=2} \frac{5r^4}{4}\sin\theta\, d\theta\, d\phi - \iint_{r=1}\frac{5r^4}{4}\sin\theta\, d\theta\, d\phi$$

$$= \frac{5(2)^4}{4}\int_0^{2\pi}\int_0^{\pi}\sin\theta\, d\theta\, d\phi - \frac{5(1)^4}{4}\int_0^{2\pi}\int_0^{\pi}\sin\theta\, d\theta\, d\phi$$

$$= 20\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi} d\phi - \frac{5}{4}\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi} d\phi$$

$$= 40\left[\phi\right]_0^{2\pi} - \frac{5}{2}\left[\phi\right]_0^{2\pi}$$

$$= 80\pi - 5\pi$$

$$\iint \vec{D}\cdot\vec{ds} = 75\pi \quad \text{—————} (2)$$

from equation (1) & (2)

$$\iint \vec{D}\cdot\vec{ds} = \iiint \nabla\cdot\vec{D}\, dV$$

Case (ii): $\theta = 0$ to $\frac{\pi}{4}$ & $r = 4$

LHS : $\iiint_V \nabla \cdot \vec{D}\, dv$

$$\iiint_V \nabla\cdot\vec{D}\, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\int_{r=0}^{4} 5r^3 \sin\theta\, dr\, d\theta\, d\phi$$

$\because \nabla\cdot\vec{D} = 5r$

$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$= 5\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^4 \sin\theta\, d\theta\, d\phi$$

$$= 5\,\frac{(4)^4}{4}\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi/4} d\phi$$

$$= 320\int_0^{2\pi}\left[-\cos\frac{\pi}{4} + \cos 0\right] d\phi$$

$$= 320 \times 0.293\, \left[\phi\right]_0^{2\pi}$$

$$= 320 \times 0.293 \times 2\pi$$

$$\iiint_V \nabla\cdot\vec{D}\, dv = 187.45\pi \quad \text{------------} \quad (1)$$

RHS : $\iint_S \vec{D}\cdot\vec{ds}$

$$\iint_S \vec{D}\cdot\vec{ds} = \iint_{r=4} \vec{D}\cdot ds_r\, \vec{a_r} + \iint \vec{D}\cdot ds_\theta\, \vec{a_\theta}$$

$$= \iint_{r=4} \frac{5r^4}{4}\sin\theta\, d\theta\, d\phi + 0$$

$$= \frac{5(4)^4}{4}\int_0^{2\pi}\int_0^{\pi/4} \sin\theta\, d\theta\, d\phi = 320\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi/4} d\phi$$

$$= 320 \times \left[-\cos\frac{\pi}{4} + \cos 0\right]\int_0^{2\pi} d\phi$$

$$= 320 \times 0.293 \times 2\pi$$

$$\iint \vec{D}\cdot\vec{ds} = 187.45\pi \quad \text{------------} \quad (2)$$

from (1) & (2)

$$\iint \vec{D}\cdot\vec{ds} = \iiint_V \nabla\cdot\vec{D}\, dv.$$

(9) Given that $\vec{D} = \frac{10\rho^3}{4}\vec{a_\rho}$ in Cylindrical Coordinate System. Evaluate both sides of divergence theorem for the volume enclosed with $\rho = 2$ and $z = 0$ & $z = 10$.

Given:

Solution: $\vec{D} = \frac{10\rho^3}{4}\vec{a_\rho}$

By divergence theorem

$$\iiint_V \nabla \cdot \vec{D}\, dv = \iint_S \vec{D} \cdot \vec{ds}$$

LHS: $\iiint_V \nabla \cdot \vec{D}\, dv$

$$\nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \times \frac{10\rho^3}{4}\right) = \frac{1}{\rho} \times \frac{10}{4} \times 4\rho^3$$

$$\nabla \cdot \vec{D} = 10\rho^2$$

$$\iiint_V \nabla \cdot \vec{D}\, dv = \int_{z=0}^{10}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{2} 10\rho^2 \times \rho\, d\rho\, d\phi\, dz \qquad \because dv = \rho\, d\rho\, d\phi\, dz$$

$$= 10\int_0^{10}\int_0^{2\pi}\left[\frac{\rho^4}{4}\right]_0^2 d\phi\, dz$$

$$= 10 \times \frac{16}{4}\int_0^{10}\left[\phi\right]_0^{2\pi} dz$$

$$= 40 \times 2\pi \left[z\right]_0^{10}$$

$$= 40 \times 2\pi \times 10$$

$$\iiint_V \nabla \cdot \vec{D}\, dv = 800\pi \longrightarrow (1)$$

RHS: $\oint\!\!\oint \vec{D} \cdot \vec{ds}$

$$\oint\!\!\oint \vec{D} \cdot \vec{ds} = \iint_{\rho=2} \vec{D} \cdot ds_\rho \vec{a_\rho} + \iint \vec{D} \cdot ds_z \vec{a_z}$$

$$= \iint_{\rho=2} \frac{10\rho^3}{4} \times \rho\, d\phi\, dz + 0$$

$$= \frac{10}{4}(2)^4 \int_0^{10}\int_0^{2\pi} d\phi\, dz$$

$$= 40 \int_0^{10} [\phi]_0^{2\pi} dz$$

$$= 40 \times 2\pi \, [z]_0^{10}$$

$$= 40 \times 2\pi \times 10$$

$$\iint \vec{D} \cdot \vec{ds} = 800\pi \quad \text{———} \quad (2)$$

from (1) & (2)

$$\iiint_V \nabla \cdot \vec{D} \, dV = \iint_S \vec{D} \cdot \vec{ds}.$$

(3) For the field $\vec{D} = 2xy\,\vec{a_x} + x^2\vec{a_y}$ C/m² and the rectangular parallelepiped formed by the planes $x=0$, $x=1$, $y=0$, $y=2$, $z=0$, $z=3$

Given: $\vec{D} = 2xy\,\vec{a_x} + x^2\vec{a_y}$

Solution:

By divergence theorem

$$\iiint_V \nabla \cdot \vec{D} \, dV = \iint_S \vec{D} \cdot \vec{ds}.$$

LHS: $\iiint_V \nabla \cdot \vec{D} \, dV$

$$\nabla \cdot \vec{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial z}(0) = 2y$$

$$\iiint_V \nabla \cdot \vec{D} \, dv = \int_0^3 \int_0^2 \int_0^1 2y \, dx\, dy\, dz \qquad \because dv = dx\,dy\,dz$$

$$= 2 \int_0^3 \int_0^2 y\,(x)_0^1 \, dy\, dz$$

$$= 2(1) \int_0^3 \left(\frac{y^2}{2}\right)_0^2 dz$$

$$= \cancel{2} \times \frac{4}{\cancel{2}} [z]_0^3$$

$$= 4 \times 3$$

$$= 12 \quad \text{———} \quad (1)$$

RHS: $\oint\!\!\oint \vec{D}\cdot\vec{ds}$

$$\iint_S \vec{D}\cdot\vec{ds} = \iint_{x=0} \vec{D}\cdot ds_x(-\vec{a_x}) + \iint_{x=1} \vec{D}\cdot ds_x(\vec{a_x})$$

$$+ \iint_{y=0} \vec{D}\cdot ds_y(-\vec{a_y}) + \iint_{y=2} \vec{D}\cdot ds_y(\vec{a_y}) \ldots$$

$$+ \iint_{z=0} \vec{D}\cdot ds_z(-\vec{a_z}) + \iint_{z=3} \vec{D}\cdot ds_z(\vec{a_z})$$

$$\iint_{x=0} \vec{D}\cdot ds_x(-\vec{a_x}) = -\iint_{x=0} 2xy\,dy\,dz = 0 \quad \text{——— (A)}$$

$$\iint_{x=1} \vec{D}\cdot ds_x(\vec{a_x}) = \iint_{x=1} 2xy\,dy\,dz = 2\int_0^3\int_0^2 y\,dy\,dz = 2\int_0^3 \left(\frac{y^2}{2}\right)_0^2 dz$$

$$= \cancel{2}\times\frac{4}{\cancel{2}}(z)_0^3 = 12 \quad \text{——— (B)}$$

$$\iint_{y=0} \vec{D}\cdot ds_y(-\vec{a_y}) = -\iint_{y=0} x^2\,dx\,dz = -\int_0^3\int_0^1 x^2\,dx\,dz = -\int_0^3 \left(\frac{x^3}{3}\right)_0^1 dz$$

$$= -\frac{1}{3}(z)_0^3 = -\frac{1}{3}\times 3 = -1 \quad \text{——— (C)}$$

$$\iint_{y=2} \vec{D}\cdot ds_y(+\vec{a_y}) = \iint_{y=2} x^2\,dx\,dz = \int_0^3\int_0^1 x^2\,dx\,dz = \int_0^3 \left(\frac{x^3}{3}\right)_0^1 dz$$

$$= \frac{1}{3}(z)_0^3 = \frac{1}{3}\times 3 = 1 \quad \text{——— (D)}$$

$$\iint_{z=0} \vec{D}\cdot ds_z(-\vec{a_z}) = \iint_{z=0} 0 = 0 \quad \text{——— (E)}$$

$$\iint_{z=3} \vec{D}\cdot ds_z(\vec{a_z}) = \iint_{z=3} 0 = 0 \quad \text{——— (F)}$$

frm (A), (B), (C), (D), (E) & (F)

$$\iint \vec{D}\cdot\vec{ds} = 0+12-1+1+0+0$$

$$= 12 \quad \text{——— (2)}$$

frm (1) & (2)

$$\iiint_V \nabla\cdot\vec{D}\,dv = \iint_S \vec{D}\cdot\vec{ds}.$$

STOKE'S THEOREM:

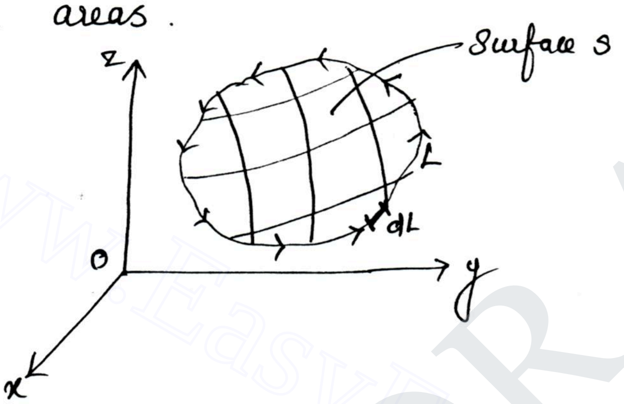
STATEMENT:

The Circulation of a vector field $\vec{A}$ around a closed path L is equal to the surface integral of the curl of $\vec{A}$ over the open surface S bounded by L, provided $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on S.

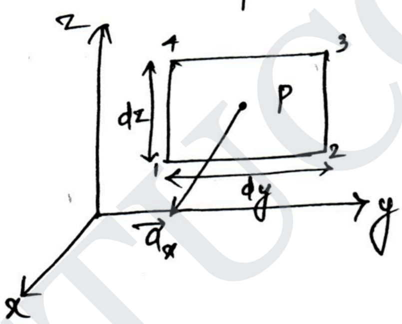
$$\oint \vec{A} \cdot \vec{dl} = \iint_S (\nabla \times \vec{A}) \cdot \vec{dS}$$

PROOF:

- Consider a arbitrary surface S.
- Subdivide the surface S into set of small approximate rectangular areas.



- Consider a separate rectangular area.



- Consider a vector $\vec{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}$

From the figure.

$$\oint \vec{A} \cdot \vec{dL} = \int_1^2 \vec{A} \cdot \vec{dL} + \int_2^3 \vec{A} \cdot \vec{dL} + \int_3^4 \vec{A} \cdot \vec{dL} + \int_4^1 \vec{A} \cdot \vec{dL}$$

for the variation in x

$$\int \vec{A} \cdot \vec{dL} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dy\, dz \quad \text{—— (1)}$$

for the variation in y

$$\int \vec{A} \cdot \vec{dL} = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) dx\, dz \quad \text{—— (2)}$$

 for the variation in z

$$\int \vec{A}\cdot\vec{dl} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy \quad \text{—— (3)}$$

There is a cancellation on every interior path, so sum of the line integrals is same as the line integral around the boundary curve L.

from (1), (2) & (3)

$$\oint \vec{A}\cdot\vec{dl} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dy\,dz + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) dx\,dz + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy \quad \text{—— (A)}$$

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \vec{a_x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \vec{a_y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \vec{a_z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{ds} = ds_x\,\vec{a_x} + ds_y\,\vec{a_y} + ds_z\,\vec{a_z} = dy\,dz\,\vec{a_x} + dx\,dz\,\vec{a_y} + dx\,dy\,\vec{a_z}$$

$$(\nabla\times\vec{A})\cdot\vec{ds} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dy\,dz + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) dx\,dz + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy$$

for the sum of all the rectangular surfaces

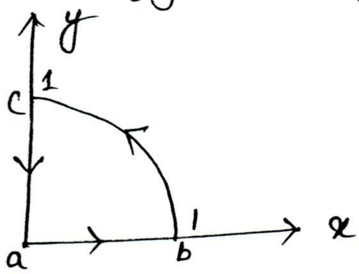
$$\iint (\nabla\times\vec{A})\cdot\vec{ds} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) dy\,dz + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) dx\,dz + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy \quad \text{—— (B)}$$

from (A) & (B)

$$\oint \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{ds}.$$

 PROBLEM:

(1) Given $\vec{A} = 2\rho\cos\phi\,\vec{a_\rho} + \rho^2\vec{a_\phi}$ in cylindrical coordinate system for a contour shown in figure. Verify Stoke's theorem.



Given: $\vec{A} = 2\rho\cos\phi\,\vec{a_\rho} + \rho^2\vec{a_\phi}$

By Stoke's theorem,

$$\oint_L \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{ds}$$

RHS: $\iint (\nabla\times\vec{A})\cdot\vec{ds}$

$$\nabla\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \vec{a_\rho} & \rho\vec{a_\phi} & \vec{a_z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \vec{a_\rho} & \rho\vec{a_\phi} & \vec{a_z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ 2\rho\cos\phi & \rho^2 & 0 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\vec{a_\rho}(0-0) - \rho\vec{a_\phi}(0-0) + \vec{a_z}(2\rho + 2\rho\sin\phi)\right]$$

$$= \frac{1}{\cancel{\rho}}\times\cancel{\rho}(2+2\sin\phi)\,\vec{a_z}$$

$$\nabla\times\vec{A} = (2+2\sin\phi)\,\vec{a_z}$$

$$\therefore \vec{ds} = \rho\, d\rho\, d\phi\, \vec{a_z}$$

$$(\nabla\times\vec{A})\cdot\vec{ds} = 2(1+\sin\phi)\rho\, d\rho\, d\phi$$

$$\iint_S (\nabla\times\vec{A})\cdot\vec{ds} = \int_0^{\pi/2}\int_0^1 2(1+\sin\phi)\rho\, d\rho\, d\phi = 2\int_0^{\pi/2}(1+\sin\phi)\left(\frac{\rho^2}{2}\right)_0^1 d\phi$$

$$= \cancel{2}\times\frac{1}{\cancel{2}}\left[\phi - \cos\phi\right]_0^{\pi/2} = \left(\frac{\pi}{2} - \cos\frac{\pi}{2}\right) - (0-\cos 0)$$

$$= \left(\frac{\pi}{2} - 0\right) - (0-1)$$

$$= 1 + \frac{\pi}{2} \quad \text{——— (1)}$$

LHS: $\oint \vec{A}\cdot\vec{dL} = \int_{AB} \vec{A}\cdot\vec{dL} + \int_{BC} \vec{A}\cdot\vec{dL} + \int_{CA} \vec{A}\cdot\vec{dL}$

for the path AB:

$\vec{dl} = d\rho\, \vec{a_\rho}$

$\phi = 0$ $\qquad$ $\vec{A}\cdot\vec{dl} = 2\rho\cos\phi\, d\rho$

$\rho \Rightarrow 0$ to 1

$$\int_{AB} \vec{A}\cdot\vec{dl} = \int_{\phi=0} 2\rho\cos\phi\, d\rho = \int_0^1 2\rho\, d\rho = 2\left(\frac{\rho^2}{2}\right)_0^1 = 2\left(\frac{1}{2}\right) = 1 \quad \text{—— (A)}$$

for the path BC:

$\vec{dl} = \rho\, d\phi\, \vec{a_\phi}$

$\rho = 1$ $\qquad$ $\vec{A}\cdot\vec{dl} = \rho^3 d\phi$

$\phi \rightarrow 0$ to $\frac{\pi}{2}$

$$\int_{BC} \vec{A}\cdot\vec{dl} = \int_{\rho=1} \rho^3 d\phi = \int_0^{\pi/2} d\phi = [\phi]_0^{\pi/2} = \frac{\pi}{2} \quad \text{—— (B)}$$

for the path CA:

$\vec{dl} = d\rho\, \vec{a_\rho}$

$\phi = \pi/2$ $\qquad$ $\vec{A}\cdot\vec{dl} = 2\rho\cos\phi\, d\rho$

$\rho \Rightarrow 1$ to 0

$$\int_{CA} \vec{A}\cdot\vec{dl} = \int_{\phi=\pi/2} 2\rho\cos\phi\, d\rho = \int_1^0 0 = 0 \quad \text{—— (C)}$$

frm (A), (B) & (C)

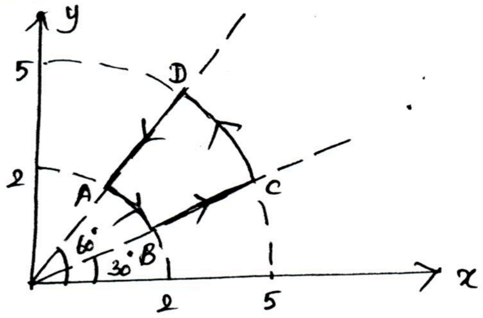
$$\oint \vec{A}\cdot\vec{dl} = 1 + \frac{\pi}{2} \quad \text{—— (2)}$$

frm (1) & (2)

$$\oint \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{ds}$$

Hence Verified.

(2) If $\vec{A} = \rho\cos\phi\,\vec{a_\rho} + \sin\phi\,a_\phi$. Verify the Stoke's theorem for the given Contour.



Given: $\vec{A} = \rho\cos\phi\,\vec{a_\rho} + \sin\phi\,\vec{a_\phi}$

By Stoke's theorem

$$\oint \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{ds}$$

RHS: $\iint_S (\nabla\times\vec{A})\cdot\vec{ds}$

$$\nabla\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \vec{a_\rho} & \rho\vec{a_\phi} & \vec{a_z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ \rho\cos\phi & \rho\sin\phi & 0 \end{vmatrix} = \frac{1}{\rho}\left[\vec{a_\rho}(0-0) - \rho\vec{a_\phi}(0-0) + \vec{a_z}(\sin\phi + \rho\sin\phi)\right]$$

$$= \frac{1}{\rho}\sin\phi(1+\rho)\,\vec{a_z}$$

$$\nabla\times\vec{A} = \sin\phi\left(1+\frac{1}{\rho}\right)\vec{a_z}$$

$$\therefore \vec{ds} = \rho\, d\rho\, d\phi\, \vec{a_z}$$

$$(\nabla\times\vec{A})\cdot\vec{ds} = \sin\phi(1+\rho)\,d\rho\,d\phi$$

$$\iint(\nabla\times\vec{A})\cdot\vec{ds} = \int_{30^\circ}^{60^\circ}\int_2^5 (1+\rho)\sin\phi\,d\rho\,d\phi = \int_{30^\circ}^{60^\circ}\left(\rho+\frac{\rho^2}{2}\right)_2^5 \sin\phi\,d\phi$$

$$= \left[\left(5+\frac{25}{2}\right)-(2+2)\right]\left[-\cos\phi\right]_{30^\circ}^{60^\circ}$$

$$= \frac{27}{2}\left[-\cos 60^\circ + \cos 30^\circ\right] = \frac{27}{2}\left[\frac{-1}{2}+\frac{\sqrt{3}}{2}\right]$$

$$= 4.941 \quad \text{——————} \quad (1)$$

LHS: $\oint \vec{A}\cdot\vec{dl} = \int_{AB} \vec{A}\cdot\vec{dl} + \int_{BC} \vec{A}\cdot\vec{dl} + \int_{CD} \vec{A}\cdot\vec{dl} + \int_{DA} \vec{A}\cdot\vec{dl}$

for the path AB:

$\vec{dl} = \rho\, d\phi\, \vec{a_\phi}$

$\rho = 2$

$\phi \to 60^\circ$ to 30°

$\vec{A}\cdot\vec{dl} = \rho \sin\phi\, d\phi = 2\sin\phi\, d\phi$

$$\int_{AB} \vec{A}\cdot\vec{dl} = \int_{60^\circ}^{30^\circ} 2\sin\phi\, d\phi = 2\left[-\cos\phi\right]_{60^\circ}^{30^\circ} = 2(-\cos 30^\circ + \cos 60^\circ) = -0.732 \quad \text{(A)}$$

for the path BC:

$\vec{dl} = d\rho\, \vec{a_\rho}$

$\phi = 30^\circ$

$\rho \to 2$ to 5

$\vec{A}\cdot\vec{dl} = \rho\cos\phi\, d\rho = \rho\cos 30^\circ\, d\rho = \frac{\sqrt{3}}{2}\rho\, d\rho.$

$$\int_{BC} \vec{A}\cdot\vec{dl} = \int_2^5 \frac{\sqrt{3}}{2}\rho\, d\rho = \frac{\sqrt{3}}{2}\left(\frac{\rho^2}{2}\right)_2^5 = \frac{\sqrt{3}}{2}\left(\frac{25-4}{2}\right) = 9.093 \quad \text{(B)}$$

for the path CD:

$\vec{dl} = \rho\, d\phi\, \vec{a_\phi}$

$\rho = 5$

$\phi \to 30^\circ$ to 60°

$\vec{A}\cdot\vec{dl} = \rho\sin\phi\, d\phi = 5\sin\phi\, d\phi$

$$\int_{CD} \vec{A}\cdot\vec{dl} = \int_{30^\circ}^{60^\circ} 5\sin\phi\, d\phi = 5\left[-\cos\phi\right]_{30^\circ}^{60^\circ} = 5\left[-\cos 60^\circ + \cos 30^\circ\right] = 1.83 \quad \text{(C)}$$

for the path DA:

$\vec{dl} = d\rho\, \vec{a_\rho}$

$\phi = 60^\circ$

$\rho \to 5$ to 2

$\vec{A}\cdot\vec{dl} = \rho\cos\phi\, d\rho = \rho\cos 60^\circ\, d\rho = \frac{\rho}{2}\, d\rho$

$$\int_{DA} \vec{A}\cdot\vec{dl} = \int_5^2 \frac{\rho}{2}\, d\rho = \frac{1}{2}\left(\frac{\rho^2}{2}\right)_5^2 = \frac{1}{2}\left(\frac{4-25}{2}\right) = -5.25 \quad \text{(D)}$$

fm (A), (B), (C) & (D)

$$\oint \vec{A}\cdot\vec{dl} = -0.732 + 9.093 + 1.83 - 5.25 = 4.941 \quad \text{(2)}$$

fm (1) & (2)

$$\oint \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{ds}$$

Hence verified.

COULOMB'S LAW:

The force between any two point charges

(i) directly proportional to product of the two charges

(ii) Inversely proportional to the square of the distance between them.

(iii) directed along the line joining two charges.

$$Q_1 \bullet \xrightarrow[\vec{r}_{12}]{\vec{u}_{12}} \bullet Q_2$$

$$\vec{F}_{12} \propto \frac{Q_1 Q_2}{r_{12}^2}\vec{u}_{12}$$

$\vec{r}_{12}$ – distance vector

$\vec{u}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}$ = Unit vector.

$$\vec{F}_{12} = k\frac{Q_1 Q_2}{r_{12}^2}\vec{u}_{12}$$

where k – Proportionality Constant.

$$k = \frac{1}{4\pi\varepsilon} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}$$

ε_0 – Absolute permittivity $= 8.854\times10^{-12}$ F/m.

ε_r – Relative Permittivity.

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0\varepsilon_r r_{12}^2}\vec{u}_{12}$$

$$F_{12} = |\vec{F}_{12}| = \frac{Q_1 Q_2}{4\pi\varepsilon_0\varepsilon_r r_{12}^2}.$$

(i) If the distance between the two point charges are given

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0\varepsilon_r r^2}$$

(ii) If locations of point charges are given

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0\varepsilon_r r_{12}^2}\vec{u}_{12}$$

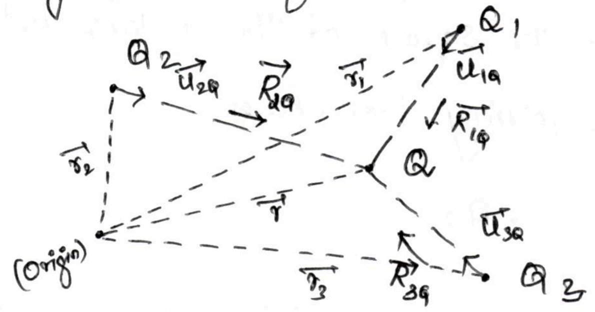
$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$$

$\vec{r}_2$ – position vector at pt of Q_2

$\vec{r}_1$ – position vector at pt of Q_1

PRINCIPLE OF SUPERPOSITION:

- If there are more than two point charges, then each will exert force on each other, then the net force on any charge can be obtained by principle of superposition.



Force exerted on Q due to Q_1 } $\vec{F}_{Q_1Q} = \dfrac{Q_1 Q}{4\pi\epsilon R_{1Q}^2}\vec{u}_{1Q}$ (Effect of Q_2 & Q_3 to be suppressed)

$$\text{where } \vec{u}_{1Q} = \frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|}$$

Force exerted on Q due to Q_2 } $\vec{F}_{Q_2Q} = \dfrac{Q_2 Q}{4\pi\epsilon R_{2Q}^2}\vec{u}_{2Q}$ (Effect of Q_1 & Q_3 to be suppressed)

$$\text{where } \vec{u}_{2Q} = \frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|}$$

Force exerted on Q due to Q_3 } $\vec{F}_{Q_3Q} = \dfrac{Q_3 Q}{4\pi\epsilon R_{3Q}^2}\vec{u}_{3Q}$ (Effect of Q_1 & Q_2 to be suppressed)

$$\text{where, } \vec{u}_{3Q} = \frac{\vec{r}-\vec{r}_3}{|\vec{r}-\vec{r}_3|}$$

The total force on Q is vector sum of all the forces exerted on Q due to each of the other point charges Q_1, Q_2 & Q_3

$$\vec{F} = \vec{F}_{Q_1Q} + \vec{F}_{Q_2Q} + \vec{F}_{Q_3Q}$$

$$= \frac{Q_1 Q}{4\pi\epsilon R_{1Q}^2}\vec{u}_{1Q} + \frac{Q_2 Q}{4\pi\epsilon R_{2Q}^2}\vec{u}_{2Q} + \frac{Q_3 Q}{4\pi\epsilon R_{3Q}^2}\vec{u}_{3Q}$$

$$= \frac{Q}{4\pi\epsilon}\left[\frac{Q_1}{R_{1Q}^2}\vec{u}_{1Q} + \frac{Q_2}{R_{2Q}^2}\vec{u}_{2Q} + \frac{Q_3}{R_{3Q}^2}\vec{u}_{3Q}\right]$$

$$\vec{F} = \frac{Q}{4\pi\epsilon}\sum_{i=1}^{3}\left[\frac{Q_i}{R_{iQ}^2}\vec{u}_{iQ}\right]$$

The force exerted on Q due to n charges

$$\vec{F} = \frac{Q}{4\pi\epsilon}\sum_{i=1}^{n}\left[\frac{Q_i}{R_{iQ}^2}\vec{u}_{iQ}\right].$$

PROBLEM:

① Find the force of interaction between two charges spaced 10 cm apart in vacuum. The charges are 4×10^{-8} C and 6×10^{-5} C. If the same charges separated by the same distance in kerosene $(\epsilon_r = 2)$. What is the force of interaction between them?

Given: $Q_1 = 4\times10^{-8}$ C $\epsilon_r = 1$ for vacuum

$Q_2 = 6\times10^{-5}$ C $\epsilon_r = 2$ for kerosene

$r = 10\text{ cm} = 10\times10^{-2}\text{ m} = 0.1\text{ m}$

Solution:

for Vacuum:

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0\epsilon_r r^2} = \frac{4\times10^{-8}\times6\times10^{-5}}{4\times3.14\times8.854\times10^{-12}\times1\times(0.1)^2}$$

$$\boxed{F = 2.15\text{ N}}$$

for kerosene:

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0\epsilon_r r^2} = \frac{4\times10^{-8}\times6\times10^{-5}}{4\times3.14\times8.854\times10^{-12}\times2\times(0.1)^2}$$

$$\boxed{F = 1.075\text{ N}}$$

② A point charge of $10\mu C$ is located at $(1,2,3)$ and another point charge of $-3\mu C$ is located at $(3,0,2)$ in vacuum. Find the force between them.

Given: $Q_1 = 10\mu C = 10\times10^{-6}$ C

$Q_2 = -3\mu C = -3\times10^{-6}$ C

$\vec{r_1} = \vec{a_x} + 2\vec{a_y} + 3\vec{a_z}$

$\vec{r_2} = 3\vec{a_x} + 2\vec{a_z}$

$\epsilon_r = 1$

Solution

$\vec{r_{12}} = \vec{r_2} - \vec{r_1} = 2\vec{a_x} - 2\vec{a_y} - \vec{a_z}$

$r_{12} = |\vec{r_{12}}| = \sqrt{4+4+1} = 3$

$$\vec{u_{12}} = \frac{\vec{r_{12}}}{|\vec{r_{12}}|} = \frac{2\vec{a_x} - 2\vec{a_y} - \vec{a_z}}{3}$$

$$\vec{F_{12}} = \frac{Q_1 Q_2}{4\pi \varepsilon_0 \varepsilon_r r_{12}^2} \vec{u_{12}}$$

$$= \frac{(10\times10^{-6})(-3\times10^{-6})}{4\times3.14\times8.854\times10^{-12}\times1\times(3)^2}\left[\frac{2\vec{a_x} - 2\vec{a_y} - \vec{a_z}}{3}\right]$$

$$\vec{F_{12}} = -0.02\vec{a_x} + 0.02\vec{a_y} - 0.01\vec{a_z}$$

$$F_{12} = |\vec{F_{12}}| = 0.03 \text{ N}.$$

(3) In xy plane $Q_1 = 100\mu C$ at $(2, 3, 1)$ m experiences a repulsive force of 7.5 N because of Q_2 at $(10, 6, 5)$ m. Find Q_2.

Given:

$F = 7.5$ N $\vec{r_1} = 2\vec{a_x} + 3\vec{a_y} + \vec{a_z}$

$Q_1 = 100\mu C$ $\vec{r_2} = 10\vec{a_x} + 6\vec{a_y} + 5\vec{a_z}$

Solution: $Q_2 = ?$

$$\vec{r_{12}} = 8\vec{a_x} + 3\vec{a_y} + 4\vec{a_z}$$

$$r_{12} = |\vec{r_{12}}| = \sqrt{64+36+16} = \sqrt{116} = 10.77$$

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 \varepsilon_r r_{12}^2}$$

$$Q_2 = \frac{4\pi\varepsilon_0\varepsilon_r r_{12}^2 F}{Q_1} = \frac{4\times3.14\times8.854\times10^{-12}\times1\times(10.77)^2\times7.5}{100\times10^{-6}}$$

$$\boxed{Q_2 = 967.44\mu C}$$

4) Four point charges each of $10\mu C$ are placed in free space at the points $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$ and $(0,-1,0)$ m respectively. Determine the force on a point charge $30\mu C$ located at a point $(0,0,1)$m.

Given:

$Q = 30\mu C$	$Q_1 = 10\mu C$	$Q_2 = 10\mu C$	$Q_3 = 10\mu C$	$Q_4 = 10\mu C$
$\overline{r'} = \overline{a_z}$	$\overline{r_1'} = \overline{a_x}$	$\overline{r_2'} = -\overline{a_x}$	$\overline{r_3'} = \overline{a_y}$	$\overline{r_4'} = -\overline{a_y}$

To find $\vec{F}_{Q_1Q}$:

$$\vec{F}_{Q_1Q} = \frac{Q_1 Q}{4\pi\varepsilon_0\varepsilon_r R_{1Q}^2}\vec{u}_{1Q}$$

$$\overline{R_{1Q}} = \overline{r'} - \overline{r_1'} = \overline{a_z} - \overline{a_x}$$

$$R_{1Q} = |r - \overline{r}| = \sqrt{2}$$

$$\overline{u_{1Q}} = \frac{\overline{a_z} - \overline{a_x}}{\sqrt{2}}$$

$$\Rightarrow \vec{F}_{Q_1Q} = \frac{(10\times10^{-6})(30\times10^{-6})}{4\times3.14\times8.854\times10^{-12}\times1\times(\sqrt{2})^2}\times\left(\frac{\overline{a_z} - \overline{a_x}}{\sqrt{2}}\right)$$

$$\boxed{\vec{F}_{Q_1Q} = 0.9533(\overline{a_z} - \overline{a_x})\ N}$$

To find $\vec{F}_{Q_2Q}$:

$$\vec{F}_{Q_2Q} = \frac{Q_2 Q}{4\pi\varepsilon_0\varepsilon_r R_{2Q}^2}\vec{u}_{2Q}$$

$$\overline{R_{2Q}} = \overline{r'} - \overline{r_2'} = \overline{a_z} + \overline{a_x}$$

$$R_{2Q} = |\overline{r'} - \overline{r_2'}| = \sqrt{2}$$

$$\overline{u_{2Q}} = \frac{\overline{a_z} + \overline{a_x}}{\sqrt{2}}$$

$$\Rightarrow \vec{F}_{Q_2Q} = \frac{(10\times10^{-6})(30\times10^{-6})}{4\times3.14\times8.854\times10^{-12}\times1\times(\sqrt{2})^2}\times\left(\frac{\overline{a_z} + \overline{a_x}}{\sqrt{2}}\right)$$

$$\boxed{\vec{F}_{Q_2Q} = 0.9533(\overline{a_z} + \overline{a_x})\ N}$$

To find $\overrightarrow{F_{Q_3Q}}$:

$$\overrightarrow{F_{Q_3Q}} = \frac{Q_3 Q}{4\pi\epsilon_0 \epsilon_r R_{3Q}^2} \overrightarrow{u_{3Q}}$$

$$\overrightarrow{R_{3Q}} = \vec{r} - \vec{r_3} = \vec{a_z} - \vec{a_y}$$

$$R_{3Q} = |\vec{r} - \vec{r_3}| = \sqrt{2}$$

$$\overrightarrow{u_{3Q}} = \frac{\vec{a_z} - \vec{a_y}}{\sqrt{2}}$$

$$\overrightarrow{F_{Q_3Q}} = \frac{(10 \times 10^{-6})(30 \times 10^{-6})}{4 \times 3.14 \times 8.854 \times 10^{-12} \times 1 \times (\sqrt{2})^2} \times \left(\frac{\vec{a_z} - \vec{a_y}}{\sqrt{2}}\right)$$

$$\boxed{\overrightarrow{F_{Q_3Q}} = 0.9533 (\vec{a_z} - \vec{a_y}) \ N}$$

To find $\overrightarrow{F_{Q_4Q}}$:

$$\overrightarrow{F_{Q_4Q}} = \frac{Q_4 Q}{4\pi\epsilon_0 \epsilon_r R_{4Q}^2} \overrightarrow{u_{4Q}}$$

$$\overrightarrow{R_{4Q}} = \vec{r} - \vec{r_4} = \vec{a_z} + \vec{a_y}$$

$$R_{4Q} = |r - \vec{r_4}| = \sqrt{2}$$

$$\overrightarrow{u_{4Q}} = \frac{\vec{a_z} + \vec{a_y}}{\sqrt{2}}$$

$$\overrightarrow{F_{Q_4Q}} = \frac{(10 \times 10^{-6})(30 \times 10^{-6})}{4 \times 3.14 \times 8.854 \times 10^{-12} \times 1 \times (\sqrt{2})^2} \left(\frac{\vec{a_z} + \vec{a_y}}{\sqrt{2}}\right)$$

$$\boxed{\overrightarrow{F_{Q_4Q}} = 0.9533 (\vec{a_z} + \vec{a_y}) \ N}$$

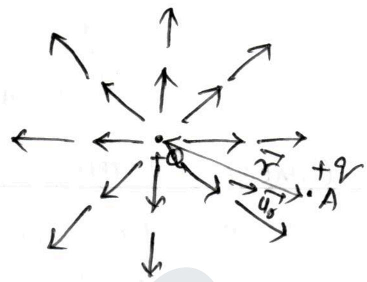
$$\vec{F} = \overrightarrow{F_{Q_1Q}} + \overrightarrow{F_{Q_2Q}} + \overrightarrow{F_{Q_3Q}} + \overrightarrow{F_{Q_4Q}}$$

$$\boxed{\vec{F} = 3.813 \, \vec{a_z} \quad N}$$

ELECTRIC FIELD INTENSITY / ELECTRIC FIELD ($\vec{E}$):

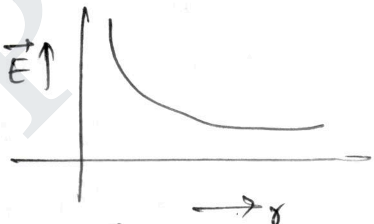
The electric field Intensity or Electric field at a point is defined as the force per unit charge on a test charge being as small as possible in comparison with other charges forming the system.

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F_q}}{q} \quad (\text{or}) \quad \frac{\vec{F}}{q}$$



where $\vec{F_q}$ – force acting on the test charge q which is small enough.

$$\vec{F_q} = \frac{Qq}{4\pi\epsilon r^2}\vec{u_r}$$

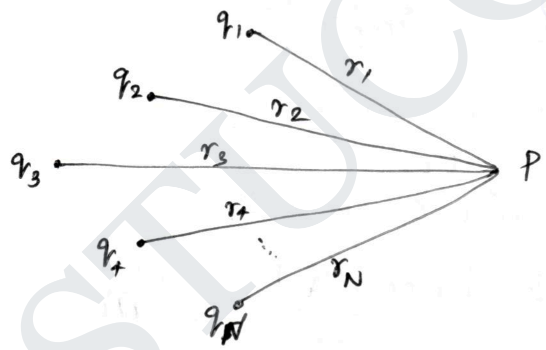


Variation of $\vec{E}$ with respect to r.

$$\Rightarrow \vec{E} = \frac{\vec{F_q}}{q} = \frac{\frac{Q\cancel{q}}{4\pi\epsilon r^2}\vec{u_r}}{\cancel{q}}$$

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon r^2}\vec{u_r}} \quad N/m \ (\text{or}) \ V/m.$$

ELECTRIC FIELD INTENSITY DUE TO ARRAY OF POINT CHARGES:



$$\vec{E} = \vec{E_1} + \vec{E_2} + \dots + \vec{E_N}$$

$$= \frac{q_1}{4\pi\epsilon r_1^2}\vec{u_1} + \frac{q_2}{4\pi\epsilon r_2^2}\vec{u_2} + \dots + \frac{q_N}{4\pi\epsilon r_N^2}\vec{u_N}$$

$$= \frac{1}{4\pi\epsilon}\left[\frac{q_1}{r_1^2}\vec{u_1} + \frac{q_2}{r_2^2}\vec{u_2} + \dots + \frac{q_N}{r_N^2}\vec{u_N}\right]$$

$$\vec{E} = \frac{1}{4\pi\epsilon}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\vec{u_i} \quad N/m \ (\text{or}) \ V/m$$

CHARGE DISTRIBUTION:

Charge Distribution.

- Point charge distribution
- Line charge distribution
- Surface charge distribution
- Volume charge distribution.

POINT CHARGE DISTRIBUTION:

- If the dimensions of a surface carrying a charge is very very small compared to region surrounding it then the surface can be treated to be a point. The corresponding charge is point charge.
- The point charge has a position but not the dimension.
- The point charge can be positive or negative.

Ex: $\overset{\bullet}{+Q}$ $\overset{\bullet}{-Q}$

LINE CHARGE DISTRIBUTION:

- If a charge is uniformly distributed along the line is called line charge.
- The line may be finite or infinite.

Ex: $- - - - - - - - -$ ρ_L $+ + + + + + +$ ρ_L

- The charge density of a line charge is ρ_L

$$\rho_L = \frac{\text{Total charge in Coulomb } (Q)}{\text{Total length in metres } (l)} \quad C/m$$

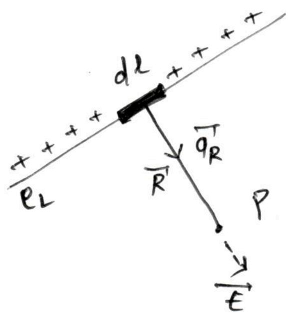
The charge dQ on the differential length dl is

$$dQ = \rho_L \, dl$$

$$Q = \int_L \rho_L \, dl .$$

- The charge distributed over a line may be positive or negative.

Electric Field Intensity due to Line Charge:



- The charge dQ on the differential length dl is

$$dQ = \rho_L \, dl$$

- The differential electric field $\vec{dE}$ at point P due to the charge dQ is

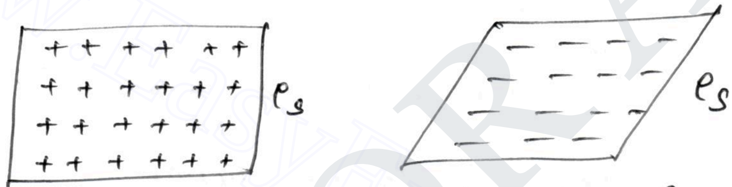
$$\vec{dE} = \frac{dQ}{4\pi\epsilon R^2}\vec{u_R} = \frac{\rho_L \, dl}{4\pi\epsilon R^2}\vec{u_R}$$

$$\vec{E} = \int_L \frac{\rho_L \, dl}{4\pi\epsilon R^2}\vec{u_R}.$$

Surface Charge Distribution:

- If the charge is distributed uniformly over a two dimensional surface then it is called as surface charge or sheet of charge.

Ex:



- The charge density of a surface charge is ρ_s

$$\rho_s = \frac{\text{Total charge in Coulomb } (Q)}{\text{Total area in Square meter } (S)} \quad C/m^2$$

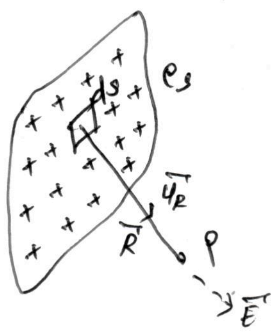
- The charge dQ on a differential area ds is

$$dQ = \rho_s \, ds$$

$$Q = \int_S \rho_s \, ds$$

- The ρ_s is constant over the surface carrying the charge.

Electric Field Intensity due to Surface Charge:



The charge dQ on the differential surface ds is

$$dQ = \rho_s \, ds$$

The differential electric field $\vec{dE}$ at point P due to dQ is

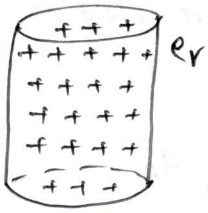
$$\vec{dE} = \frac{dQ}{4\pi\epsilon R^2}\vec{u_R} = \frac{\rho_s \, ds}{4\pi\epsilon R^2}\vec{u_R}$$

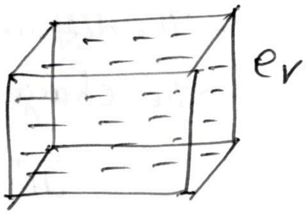
$$\vec{E} = \int_S \frac{\rho_s \, ds}{4\pi\epsilon R^2}\vec{u_R}$$

VOLUME CHARGE DISTRIBUTION:

- If the charge is distributed in a volume then it is called volume charge.

Example:





- The charge density of a volume charge is ρ_v

$$\rho_v = \frac{\text{Total charge in Coulomb } (Q)}{\text{Total volume in Cubic metres } (V)} \quad C/m^3$$

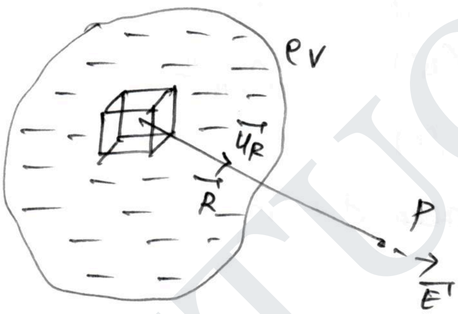
- The charge dQ on a differential volume dV is

$$dQ = \rho_v \, dV$$

$$Q = \int_V \rho_v \, dv$$

- The ρ_v is constant over the volume carrying the charge.

ELECTRIC FIELD INTENSITY DUE TO VOLUME CHARGE:



The charge dQ on a differential volume dV is

$$dQ = \rho_v \, dV$$

The differential electric field $\vec{dE}$ at point P due to dQ is

$$\vec{dE} = \frac{dQ}{4\pi\epsilon R^2} \vec{u_R}$$

$$\vec{dE} = \frac{\rho_v \, dv}{4\pi\epsilon R^2} \vec{u_R}$$

$$\vec{E} = \int_V \frac{\rho_v \, dv}{4\pi\epsilon R^2} \vec{u_R}.$$

ELECTRIC POTENTIAL

WORKDONE:

– Consider a uniform electric field $\vec{E}$ and unit positive charge (q) in the field.

– If the test charge is moved from one point to another point in the electric field, there is a force act on the test charge due to the electric field. The field is given by

$$\vec{E} = \frac{\vec{F}}{q} \Rightarrow \vec{F} = q\vec{E}$$

– There is a movement of charge in the electric field from one point to another point then there will be a workdone against the force.

Workdone = Force × distance.

$$dW = -\vec{F} \cdot \vec{dr}$$

– Ve sign indicates force and workdone are opp in directn

Sub $\vec{F} = q\vec{E}$

$$dW = -q\,\vec{E} \cdot \vec{dr}$$

$$\boxed{W = -q\int_{r_1}^{r_2} \vec{E} \cdot \vec{dr}}$$

POTENTIAL DIFFERENCE / POTENTIAL (V):

The workdone in moving a positive charge from one point to another point in a electric field.

$$V = \frac{W}{q}$$

$r_1 \longrightarrow r_2$

Sub $W = -q\int_{r_1}^{r_2} \vec{E} \cdot \vec{dr}$

$$V = \frac{-\cancel{q}\int_{r_1}^{r_2} \vec{E} \cdot \vec{dr}}{\cancel{q}}$$

$$\boxed{V = -\int \vec{E} \cdot \vec{dr}} \Rightarrow \boxed{dV = -\vec{E} \cdot \vec{dr}}$$

For a point charge $\vec{E} = \frac{Q}{4\pi\epsilon r^2} u_r$

$$\vec{dr} = dr\,\vec{u_r}$$

$$\vec{E}\cdot\vec{dr} = \frac{Q}{4\pi\epsilon r^2}\,dr$$

$$\therefore V = -\int_{r_1}^{r_2} \frac{Q}{4\pi\epsilon r^2}\,dr$$

$$= \frac{-Q}{4\pi\epsilon}\int_{r_1}^{r_2} \frac{1}{r^2}\,dr$$

$$= \frac{-Q}{4\pi\epsilon}\left[\frac{-1}{r}\right]_{r_1}^{r_2}$$

$$= \frac{-Q}{4\pi\epsilon}\left[\frac{-1}{r_2} + \frac{1}{r_1}\right]$$

$$= \frac{Q}{4\pi\epsilon r_2} - \frac{Q}{4\pi\epsilon r_1}$$

$$V = V_2 - V_1$$

In general

$$\boxed{V = \frac{Q}{4\pi\epsilon r}} \text{ Volts}$$

ABSOLUTE POTENTIAL:

The Workdone in moving the unit positive charge from infinite point to a given point in a uniform electric field is known as absolute potential.

$$V = \frac{Q}{4\pi\epsilon r} \text{ Volts.}$$

CONSERVATIVE FIELD:

The field in which the closed line integral of the field is equal to zero is said to be Conservative field.

$$\oint \vec{E} \cdot \vec{dr} = 0$$

The potential difference is not independant of path, but in Conservative field the workdone in moving unit +ve charge from one point to another point is independant of path.

RELATIONSHIP BETWEEN ELECTRIC FIELD INTENSITY (E) AND POTENTIAL (V):

The potential V can be written as

$$V = -\int \vec{E} \cdot \vec{dr}$$

Differentiate on both sides

$$dV = -\vec{E} \cdot \vec{dr}$$

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = -\vec{E} \cdot \vec{dr}$$

$$\left(\frac{\partial V}{\partial x} \vec{a_x} + \frac{\partial V}{\partial y} \vec{a_y} + \frac{\partial V}{\partial z} \vec{a_z}\right) \cdot \left(dx\, \vec{a_x} + dy\, \vec{a_y} + dz\, \vec{a_z}\right) = -\vec{E} \cdot \vec{dr}$$

$$(\nabla V) \cdot \vec{dr} = -\vec{E} \cdot \vec{dr}$$

$$\nabla V = -\vec{E}$$

$$\boxed{\vec{E} = -\nabla V}$$

The electric field intensity at a point is equal to the negative of the potential gradient at that point.

PROBLEM:

① For the given potential $V = x^2 y + 10yz$ find $\vec{E}$ at $(2, -1, 3)$

Given: $V = x^2 y + 10yz$

Soln/

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial x} \vec{a_x} + \frac{\partial V}{\partial y} \vec{a_y} + \frac{\partial V}{\partial z} \vec{a_z}\right]$$

$$\vec{E} = -\left[2xy\, \vec{a_x} + (x^2 + 10z)\, \vec{a_y} + 10y\, \vec{a_z}\right]$$

$$(\vec{E})_{(2,-1,3)} = -\left[-2\vec{a_x} + 34\vec{a_y} - 10\vec{a_z}\right]$$

$$\boxed{(\vec{E})_{(2,-1,3)} = 2\vec{a_x} - 34\vec{a_y} + 10\vec{a_z}}$$

(2) For the given field $\vec{E}$, find the potential difference V_{AB} between A (−7, 2, 1) and B (4, 1, 2).

$$\vec{E} = \frac{-6y}{x^2}\vec{a_x} + \frac{6}{x}\vec{a_y} + 5\vec{a_z} \ V/m.$$

Given:

$$\vec{E} = \frac{-6y}{x^2}\vec{a_x} + \frac{6}{x}\vec{a_y} + 5\vec{a_z}$$

$x \Rightarrow +4$ to -7

$y \Rightarrow 1$ to 2

$z \Rightarrow 2$ to 1

Soln/.

$$\vec{dr} = dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}$$

$$\vec{E}\cdot\vec{dr} = \frac{-6y}{x^2}dx + \frac{6}{x}dy + 5dz.$$

$$V_{AB} = -\int \vec{E}\cdot\vec{dr} = -\int\left(\frac{-6y}{x^2}dx + \frac{6}{x}dy + 5dz\right)$$

$$= -\left\{\int_{y=2} \frac{-6y}{x^2}dx + \int_{x=-7} \frac{6}{x}dy + \int 5dz\right\}$$

$$= -\left\{-12\int_{x=4}^{-7}\frac{1}{x^2}dx - \frac{6}{7}\int_1^2 dy + 5\int_2^1 dz\right\}$$

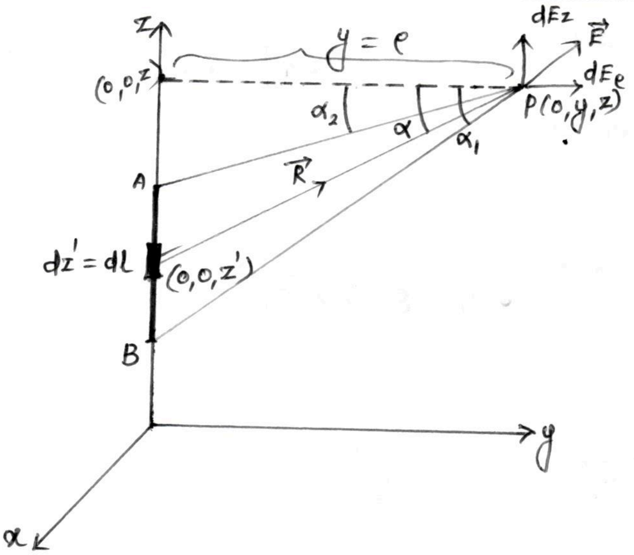
$$= -\left\{-12\left(\frac{-1}{x}\right)_4^{-7} - \frac{6}{7}(y)_1^2 + 5(z)_2^1\right\}$$

$$= -\left\{-12\left(\frac{1}{7}+\frac{1}{4}\right) - \frac{6}{7}(2-1) + 5(1-2)\right\}$$

$$= -\left\{\frac{33}{7} - \frac{6}{7} - 5\right\} = -\left\{\frac{-8}{7}\right\}$$

$$\boxed{V_{AB} = 1.14 \text{ Volts}}$$

ELECTRIC FIELD INTENSITY DUE TO FINITE AND INFINITE LINE :



- Consider a uniformly charged line with line charge density ρ_L C/m along z-axis.
- Let P be any point at which the field intensity has to be determined.
- Consider a differential length $dl = dz'$ which has a distance of R from the point P.
- The charge element dQ associated with $dl = dz'$ of the line is

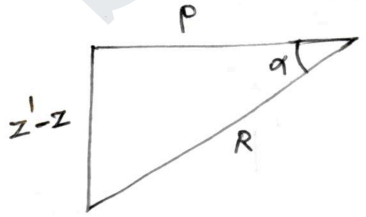
$$dQ = \rho_L dl = \rho_L dz'$$

The electric field Intensity due to line charge is

$$\vec{E} = \int_L \frac{dQ}{4\pi\varepsilon R^2}\vec{a_R} = \int_L \frac{\rho_L dz' \vec{R}}{4\pi\varepsilon R^3} \qquad \because dQ = \rho_L dz', \quad \vec{a_R} = \frac{\vec{R}}{R}$$

The distance vector, $\vec{R} = (0-0)\vec{a_x} + (y-0)\vec{a_y} + (z-z')\vec{a_z}$

$$= y\vec{a_y} + (z-z')\vec{a_z}$$



from this triangle

$$\tan\alpha = \frac{z'-z}{\rho}$$

$$z'-z = \rho\tan\alpha$$

$$\cos\alpha = \frac{\rho}{R}$$

$$R = \frac{\rho}{\cos\alpha}$$

$$\boxed{R = \rho\sec\alpha}$$

Differentiate on both sides

$$dz' - 0 = \rho\sec^2\alpha\, d\alpha$$

$$\boxed{dz' = \rho\sec^2\alpha\, d\alpha}$$

In cylindrical,

$$\vec{R} = \rho\vec{a_\rho} - \rho\tan\alpha\,\vec{a_z} \qquad \because z'-z = \rho\tan\alpha$$

$$\therefore \vec{E} = \int_L \frac{\rho_L \, \rho \sec^2\alpha \, d\alpha}{4\pi\varepsilon \, \rho^3 \sec^3\alpha} \left(\rho \vec{a}_\rho - \rho \tan\alpha \, \vec{a}_z\right)$$

$$= \int_L \frac{\rho_L \, \rho^2 \sec^2\alpha \, d\alpha}{4\pi\varepsilon \, \rho^3 \sec^3\alpha} \left(\vec{a}_\rho - \tan\alpha \, \vec{a}_z\right)$$

$$= \frac{\rho_L}{4\pi\varepsilon\rho} \int_L \cos\alpha \left(\vec{a}_\rho - \frac{\sin\alpha}{\cos\alpha} \vec{a}_z\right) d\alpha$$

for finite length

$$= \frac{\rho_L}{4\pi\varepsilon\rho} \int_{\alpha_2}^{\alpha_1} \left(\cos\alpha \, \vec{a}_\rho - \sin\alpha \, \vec{a}_z\right) d\alpha$$

$$= \frac{\rho_L}{4\pi\varepsilon\rho} \Big[\sin\alpha \, \vec{a}_\rho + \cos\alpha \, \vec{a}_z \Big]_{\alpha_2}^{\alpha_1}$$

$$= \frac{\rho_L}{4\pi\varepsilon\rho} \Big[\left(\sin\alpha_1 \vec{a}_\rho + \cos\alpha_1 \vec{a}_z\right) - \left(\sin\alpha_2 \vec{a}_\rho + \cos\alpha_2 \vec{a}_z\right) \Big]$$

$$\boxed{\vec{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \Big[\left(\sin\alpha_1 - \sin\alpha_2\right) \vec{a}_\rho + \left(\cos\alpha_1 - \cos\alpha_2\right) \vec{a}_z \Big] \quad V/m.}$$

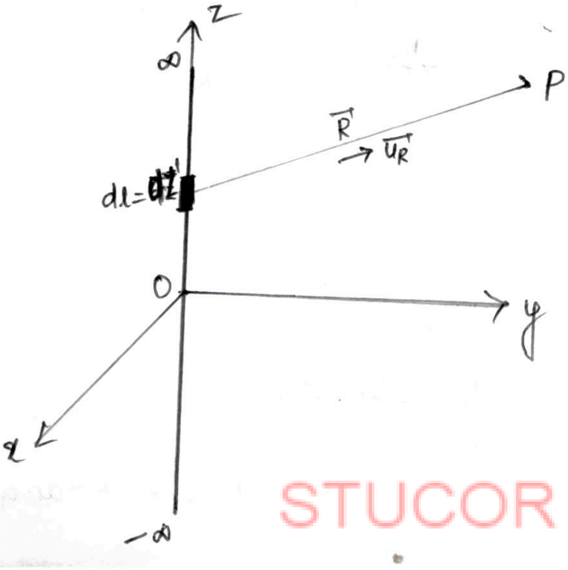
<u>for Infinite length of line :</u>

$$\alpha_1 = \frac{\pi}{2}, \quad \alpha_2 = \frac{-\pi}{2}$$

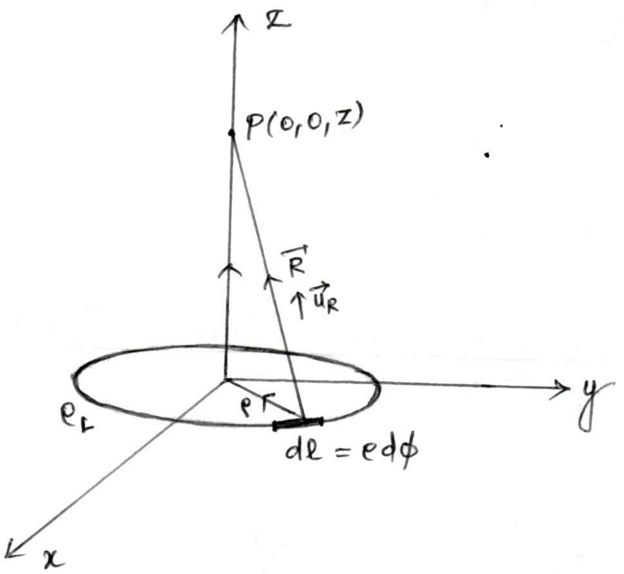
$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon\rho} \Big[\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right) \vec{a}_\rho + \left(\cos\frac{\pi}{2} - \cos\frac{\pi}{2}\right) \vec{a}_z \Big]$$

$$= \frac{\rho_L}{4\pi\varepsilon\rho} \Big[2\vec{a}_\rho \Big]$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho} \vec{a}_\rho \quad V/m.}$$



ELECTRIC FIELD INTENSITY DUE TO CIRCULAR RING



- Consider a circular ring of radius r placed in xy plane with center at origin with charge density ρ_L C/m.
- Let P be any point at which the field intensity has to be determined.
- Consider a differential length $dl = \rho\, d\phi$ which has a distance of R from the point P.

The charge dQ associated with $dl = \rho\, d\phi$ of the ring is

$$dQ = \rho_L\, dl = \rho_L\, \rho\, d\phi$$

The electric field intensity due to the line charge is

$$\vec{E} = \int_L \frac{dQ}{4\pi\epsilon R^2}\vec{u}_R = \int_L \frac{\rho_L\, \rho\, d\phi\, \vec{R}}{4\pi\epsilon R^3} \qquad \because dQ = \rho_L\, \rho\, d\phi, \quad \vec{u}_R = \frac{\vec{R}}{R}$$

$$\vec{R} = \rho(-\vec{a}_\rho) + z\vec{a}_z = -\rho\vec{a}_\rho + z\vec{a}_z$$

$$R = |\vec{R}| = \sqrt{\rho^2 + z^2}$$

$$\therefore \vec{E} = \int_L \frac{\rho_L\, \rho\, d\phi\, (-\rho\vec{a}_\rho + z\vec{a}_z)}{4\pi\epsilon\left(\sqrt{\rho^2+z^2}\right)^3}$$

The radial components are symmetrical about z-axis. So they cancel each other. So $\vec{E}$ doesn't have component in $\vec{a}_\rho$ direction.

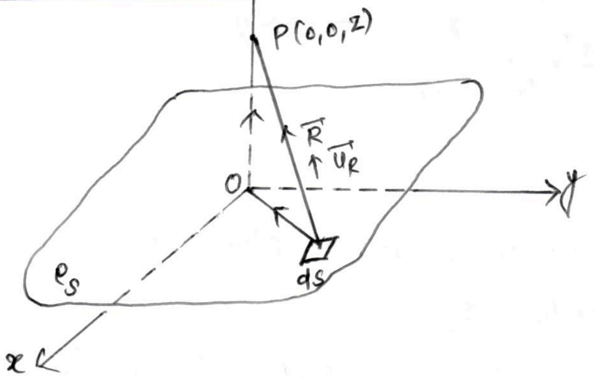
$$\vec{E} = \int_L \frac{\rho_L\, \rho\, d\phi\, (z\vec{a}_z)}{4\pi\epsilon(\rho^2+z^2)^{3/2}} = \frac{\rho_L\, \rho z}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \int_0^{2\pi} d\phi\, \vec{a}_z$$

$$= \frac{\rho_L\, \rho z}{4\pi\epsilon(\rho^2+z^2)^{3/2}} \left[\phi\right]_0^{2\pi} \vec{a}_z$$

$$\vec{E} = \frac{\rho_L \rho z}{4\pi\varepsilon(\rho^2+z^2)^{3/2}} \times 2\pi \vec{a_z}$$

$$\boxed{\vec{E} = \frac{\rho_L \rho z}{2\varepsilon(\rho^2+z^2)^{3/2}} \vec{a_z}} \quad V/m.$$

ELECTRIC FIELD INTENSITY DUE TO INFINITE SHEET OF CHARGE:



- Consider a infinite sheet of charge having uniform charge density of ρ_s C/m² placed in xy plane.
- Let P be any point at which the field intensity has to be determined.
- Consider a differential surface ds carrying charge dQ. The normal vector of ds is $\vec{a_z}$.

$$\therefore ds = \rho\, d\rho\, d\phi$$

- The charge dQ associated with $ds = \rho\, d\rho\, d\phi$ of the sheet is

$$dQ = \rho_s ds = \rho_s \rho\, d\rho\, d\phi$$

The electric field intensity due to surface charge is,

$$\vec{E} = \int_S \frac{dQ}{4\pi\varepsilon R^2} \vec{u_R} = \int_S \frac{\rho_s \rho\, d\rho\, d\phi}{4\pi\varepsilon R^3} \vec{R}$$

$$\vec{R} = -\rho \vec{a_\rho} + z\vec{a_z}$$

$$R = |\vec{R}| = \sqrt{\rho^2+z^2}$$

$$\vec{E} = \int \frac{\rho_s \rho\, d\rho\, d\phi}{4\pi\varepsilon\left(\sqrt{\rho^2+z^2}\right)^3}(-\rho\vec{a_\rho} + z\vec{a_z})$$

The radial components are symmetrical about z-axis, so they cancel each other. $\vec{E}$ doesn't have component in $\vec{a_\rho}$ direction.

$$\vec{E} = \int \frac{\rho_s \rho \, d\rho \, d\phi}{4\pi\varepsilon (\rho^2+z^2)^{3/2}} (z \vec{a_z})$$

$$= \frac{\rho_s}{4\pi\varepsilon} \int_0^{2\pi} \int_0^{\infty} \frac{\rho \, d\rho \, d\phi}{(\rho^2+z^2)^{3/2}} z \vec{a_z}$$

let $\rho^2 + z^2 = u^2$ If $\rho = 0, \; u = z$

$2\rho \, d\rho = 2u \, du$ If $\rho = \infty, \; u = \infty$

$\boxed{\rho \, d\rho = u \, du}$

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon} \int_0^{2\pi} \int_z^{\infty} \frac{u \, du}{u^3} \, d\phi \, z \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\varepsilon} \int_0^{2\pi} \int_z^{\infty} z \frac{1}{u^2} \, du \, d\phi \, \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\varepsilon} \int_0^{2\pi} z \left[\frac{-1}{u}\right]_z^{\infty} d\phi \, \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\varepsilon} \int_0^{2\pi} z \left[0 + \frac{1}{z}\right] d\phi \, \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\varepsilon} [\phi]_0^{2\pi} \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\varepsilon} \times 2\pi \, \vec{a_z}$$

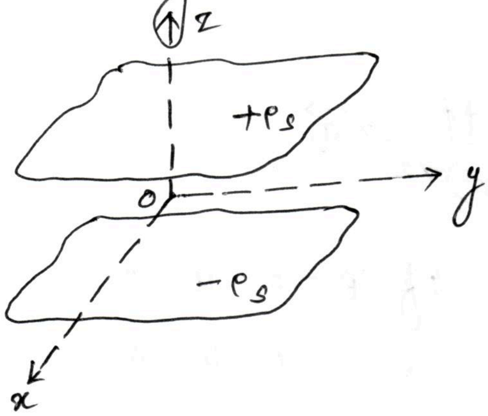
$$\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon} \vec{a_z}} \quad V/m.$$

In general

$$\vec{E} = \frac{\rho_s}{2\varepsilon} \vec{a_n} \quad V/m$$

where, $\vec{a_n} \rightarrow$ Normal vector the surface.

– If two sheets with charge densities $+\rho_s$ and $-\rho_s$ C/m^2 is located in a system,



The electric field Intensity between the sheets $\vec{E} = \vec{E_1} + \vec{E_2}$

$$= \frac{\rho_s}{2\epsilon}(\vec{a_z}) + \left(\frac{-\rho_s}{2\epsilon}\right)(-\vec{a_z})$$

$$\vec{E} = \frac{\rho_s}{\epsilon}\vec{a_z} \quad V/m$$

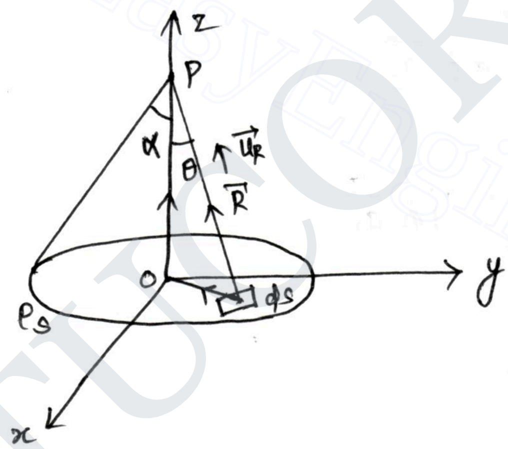
$\vec{E}$ above the sheets

$$\vec{E} = \frac{\rho_s}{2\epsilon}\vec{a_z} + \left(\frac{-\rho_s}{2\epsilon}\right)(\vec{a_z}) = 0$$

$\vec{E}$ below the sheets

$$\vec{E} = \frac{\rho_s}{2\epsilon}(-\vec{a_z}) + \left(\frac{-\rho_s}{2\epsilon}\right)(-\vec{a_z}) = 0.$$

ELECTRIC FIELD INTENSITY DUE TO A CIRCULAR DISC:



– Consider a circular disc with uniform charge density of ρ_s C/m^2 placed in xy plane.

– Let P be any point at which the field intensity has to be determined.

– Consider a differential surface ds carrying charge dQ. The normal vector of ds is $\vec{a_z}$

$$\therefore ds = \rho\, d\rho\, d\phi$$

The charge dQ associated with $ds = \rho\, d\rho\, d\phi$ of the disc is

$$dQ = \rho_s\, ds = \rho_s\, \rho\, d\rho\, d\phi$$

The electric field intensity due to surface charge is

$$\vec{E} = \int \frac{dQ}{4\pi\epsilon R^2}\vec{u_R} = \int_s \frac{\rho_s\, \rho\, d\rho\, d\phi}{4\pi\epsilon R^3}\vec{R}$$

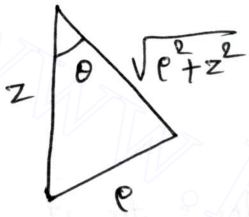
$$\vec{R} = -\rho \vec{a_\rho} + z \vec{a_z}$$

$$R = |\vec{R}| = \sqrt{\rho^2 + z^2}$$

$$\vec{E} = \int_S \frac{\rho_s \rho \, d\rho \, d\phi}{4\pi\epsilon \left(\sqrt{\rho^2+z^2}\right)^3} \left(-\rho \vec{a_\rho} + z \vec{a_z}\right)$$

The radial components are symmetrical about z-axis, so they cancel each other $\therefore \vec{E}$ doesn't have component in $\vec{a_\rho}$ direction.

$$\vec{E} = \iint \frac{\rho_s \rho \, d\rho \, d\phi}{4\pi\epsilon \left(\rho^2+z^2\right)^{3/2}} \left(z \vec{a_z}\right)$$



$$\tan\theta = \frac{\rho}{z}$$

$$\rho = z\tan\theta$$

$$d\rho = z\sec^2\theta \, d\theta.$$

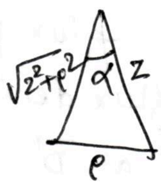
$$\vec{E} = \int_0^{2\pi}\int_0^{\alpha} \frac{\rho_s (z\tan\theta)(z\sec^2\theta \, d\theta) d\phi}{4\pi\epsilon \left(z^2\tan^2\theta + z^2\right)^{3/2}} z \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\epsilon} \int_0^{2\pi}\int_0^{\alpha} \frac{\cancel{z^3} \tan\theta \cancel{\sec^2}\theta}{\cancel{z^3}\sec^{\cancel{3}}\theta} \, d\theta \, d\phi \, \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\epsilon} \int_0^{2\pi}\int_0^{\alpha} \sin\theta \, d\theta \, d\phi \, \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\epsilon} \int_0^{2\pi} \left[-\cos\theta\right]_0^{\alpha} d\phi \, \vec{a_z}$$

$$= \frac{\rho_s}{4\pi\epsilon} (1-\cos\alpha) \left[\phi\right]_0^{2\pi}$$



$$\boxed{\vec{E} = \frac{\rho_s}{2\epsilon}(1-\cos\alpha)\vec{a_z}}$$

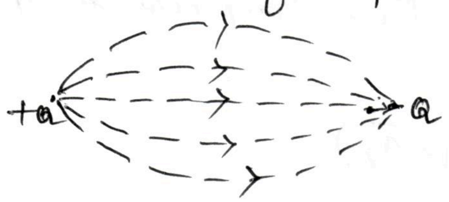
where $\cos\alpha = \dfrac{z}{\sqrt{z^2+\rho^2}}$

ELECTRIC FLUX:

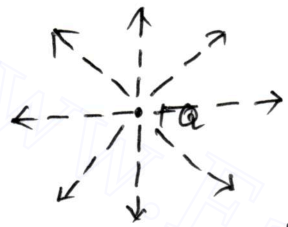
- The total number of lines of force in any particular electric field is called flux.
- Represented by the symbol ψ.
- Unit is Coulomb.

PROPERTIES:

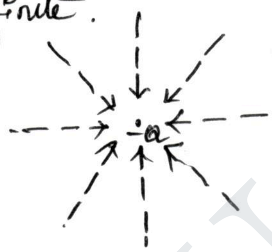
(i) Flux lines starts from positive charge and terminates at -ve charge.



(ii) If there is no -ve charge, the flux lines terminates at infinite.



(iii) If there is no +ve charge, the flux lines terminates at -ve charge from infinite.



(iii) If no. of flux lines are more, the electric field is stronger.

(iv) Flux lines are parallel, they never cross each other.

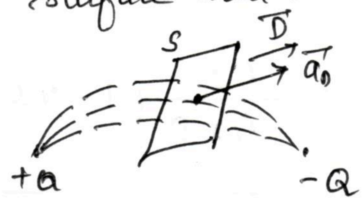
(v) Flux lines are independant of the medium.

(vi) Flux lines enter or leave the charged surface normally.

(vii) No. of flux lines is equal to total charge. $(\psi = Q)$.

ELECTRIC FLUX DENSITY:

- The net flux passing through the unit surface area is called as electric flux density.
- Denoted as $\vec{D}$.



$$D = \frac{\psi}{S} \quad C/m^2.$$

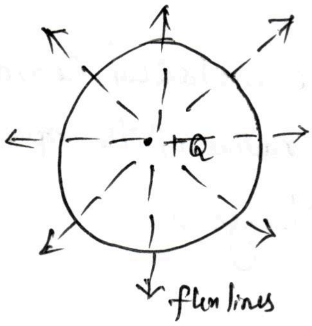
- Also called as flux density or displacement density.

In vector form

$$\vec{D} = \frac{d\psi}{dS}\vec{a_n} \quad C/m^2$$

where, $\vec{a_n}$ – Normal vector to the surface.

ELECTRIC FLUX DENSITY DUE TO POINT CHARGE:



– Consider a point charge +Q in spherical coordinate system.

– The flux lines originating from the point charge +Q are directed radially outward.

$$|\vec{D}| = \frac{\text{Total flux } \psi}{\text{Total surface area } S}$$

here $\psi = Q$, $S = 4\pi r^2$

$$|\vec{D}| = \frac{Q}{4\pi r^2}$$

$$\vec{D} = |\vec{D}|\,\vec{a_r}$$

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r}} \quad C/m^2$$

RELATION BETWEEN $\vec{D}$ AND $\vec{E}$:

$\vec{E}$ for a point charge is

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\vec{a_r} \quad \text{—— (1)}$$

$\vec{D}$ for a point charge is

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r} \quad \text{—— (2)}$$

$\frac{(1)}{(2)}$

$$\frac{\vec{E}}{\vec{D}} = \frac{(Q/4\pi\epsilon r^2)\,\vec{a_r}}{(Q/4\pi r^2)\,\vec{a_r}}$$

$$\frac{\vec{E}}{\vec{D}} = \frac{1}{\epsilon}$$

$$\epsilon\vec{E} = \vec{D}$$

$$\boxed{\vec{D} = \epsilon\vec{E}}$$

⇒ $\vec{D}$ & $\vec{E}$ are related through permittivity

$\vec{D}$ & $\vec{E}$ both act in same direction.

GAUSS LAW:

STATEMENT:

Electric flux passing through any closed surface is equal to the charge enclosed by the surface.

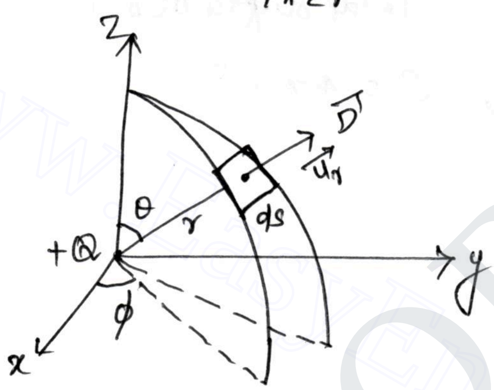
$$\boxed{\psi = Q}$$

PROOF:

Consider a charge Q at the origin of the spherical coordinate system whose coordinates are r, θ and ϕ. The radius of the sphere is r.

The electric field intensity due to point charge is

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\vec{u_r} \quad \text{--- (1)}$$



The electric flux density is

$$\vec{D} = \frac{\text{Electric flux}}{\text{Area}} = \frac{d\psi}{ds}\vec{a_n}$$

The relation between $\vec{E}$ and $\vec{D}$ is

$$\vec{D} = \epsilon\vec{E} \quad \text{--- (2)}$$

Sub eqn (1) in (2)

$$\vec{D} = \epsilon \times \frac{Q}{4\pi\epsilon r^2}\vec{u_r}$$

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{u_r} \quad \text{--- (4)}$$

$$\Rightarrow \quad d\psi = \vec{D}\cdot\vec{ds}$$

$$\psi = \int \vec{D}\cdot\vec{ds} \quad \text{--- (5)}$$

Along the radial direction, the differential area can be written as,

$$\vec{ds} = r^2 \sin\theta \, d\theta \, d\phi \, \vec{u_r}$$

$$\vec{D}\cdot\vec{ds} = \frac{Q}{4\pi \cancel{r^2}} \cancel{r^2} \sin\theta\, d\theta\, d\phi = \frac{Q}{4\pi} \sin\theta\, d\theta\, d\phi$$

$$\psi = \int_S \vec{D}\cdot\vec{ds} = \int_S \frac{Q}{4\pi} \sin\theta\, d\theta\, d\phi$$

$$= \frac{Q}{4\pi} \int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi$$

$$= \frac{Q}{4\pi} \int_0^{2\pi} [-\cos\theta]_0^{\pi}\, d\phi$$

$$= \frac{Q}{4\pi} \times 2\, [\phi]_0^{2\pi}$$

$$= \frac{Q}{2\pi} \times 2\pi$$

$$\boxed{\psi = Q}$$

Hence Proved.

INTEGRAL FORM OF GAUSS LAW:

$$\psi = \int \vec{D}\cdot\vec{ds}$$

By Gauss law $\psi = Q$

$$Q = \int \vec{D}\cdot\vec{ds}$$

$$\boxed{\int \rho_v\, dv = \int \vec{D}\cdot\vec{ds}}$$

DIFFERENTIAL OR POINT FORM OF GAUSS LAW:

The divergence theorem for $\vec{D}$

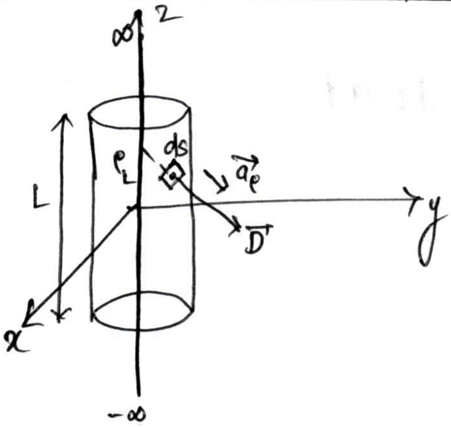
$$\int_V (\nabla\cdot\vec{D})\, dv = \int_S \vec{D}\cdot\vec{ds}$$

$$\int_V \nabla\cdot\vec{D}\, dv = \int_V \rho_v\, dv$$

$$\boxed{\nabla\cdot\vec{D} = \rho_v}$$

APPLICATIONS OF GAUSS LAW:

1. $\vec{D}$ FOR INFINITE LINE CHARGE:



- Consider an infinite line charge of density ρ_L C/m lying on z axis.
- Consider a gaussian surface as the right circular cylinder of length L

$$Q = \int \vec{D} \cdot \vec{ds}$$

$$Q = \int_{top} \vec{D} \cdot \vec{ds} + \int_{bottom} \vec{D} \cdot \vec{ds} + \int_{lateral} \vec{D} \cdot \vec{ds} \quad \text{--- (1)}$$

The direction of $\vec{D}$ is in $\vec{a_\rho}$ direction

$$\therefore \vec{D} = D\vec{a_\rho} \quad \text{--- (2)}$$

for bottom and top surface normal vector is $\vec{a_z}$. Therefore

$$\int_{top} \vec{D} \cdot \vec{ds} = 0 \quad \& \quad \int_{bottom} \vec{D} \cdot \vec{ds} = 0.$$

for lateral surface

$$\vec{ds} = \rho\, d\phi\, dz\, \vec{a_\rho}$$

$$\vec{D} \cdot \vec{ds} = D\rho\, d\phi\, dz$$

$$Q = \int_S D\rho\, d\phi\, dz = \int_0^L \int_0^{2\pi} D\rho\, d\phi\, dz$$

$$= D\rho \int_0^L [\phi]_0^{2\pi} dz$$

$$= 2\pi\rho D [z]_0^L$$

$$Q = 2\pi\rho D L$$

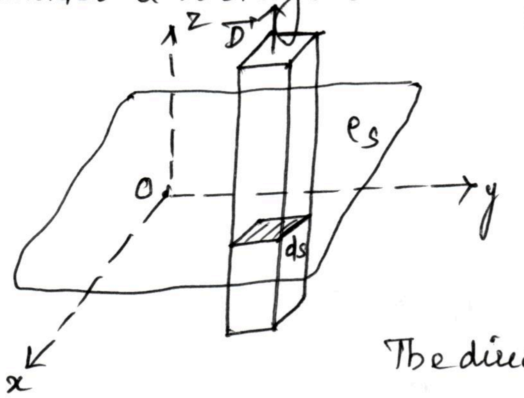
$$\boxed{D = \frac{Q}{2\pi\rho L}} \quad C/m^2$$

$$\therefore \vec{D} = \frac{Q}{2\pi\rho L}\vec{a_\rho} \; C/m^2 = \frac{\rho_L}{2\pi\rho}\vec{a_\rho} \; C/m^2 \qquad \because \rho_L = \frac{Q}{L}$$

$$\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{\rho_L}{2\pi\epsilon\rho}\vec{a_\rho} \; V/m.$$

2. $\vec{D}$ FOR INFINITE SHEET OF CHARGE :

- Consider the infinite sheet of charge of uniform charge density ρ_s C/m² lying on xy plane.
- Consider a rectangular box as gaussian surface.



$$Q = \int \vec{D} \cdot \vec{ds}$$

$$= \int_{top} \vec{D} \cdot \vec{ds} + \int_{bottom} \vec{D} \cdot \vec{ds} + \int_{sides} \vec{D} \cdot \vec{ds} \quad \text{--- (1)}$$

The direction of $\vec{D}$ is in $\vec{a_z}$ direction.

$$\therefore \vec{D} = D\,\vec{a_z} \quad \text{--- (2)}$$

The sides of a gaussian surface doesn't have the normal vector $\vec{a_z}$. So

$$\int_{sides} \vec{D} \cdot \vec{ds} = 0$$

for top surface,

$$\vec{ds} = dx\,dy\,\vec{a_z}$$

$$\therefore \vec{D} \cdot \vec{ds} = D\,dx\,dy$$

for bottom surface

$$\vec{ds} = dx\,dy\,(-\vec{a_z})$$

$$\vec{D} = D(-\vec{a_z})$$

$$\vec{D} \cdot \vec{ds} = D\,dx\,dy.$$

$$Q = \int_{top} D\,dx\,dy + \int_{bottom} D\,dx\,dy = 2D \int dx\,dy$$

$$Q = 2DA \quad \text{--- (3)} \qquad \because \text{Area}(A) = \int dx\,dy$$

WKT $\quad Q = \rho_s A \quad \text{--- (4)}$

frm (3) & (4)

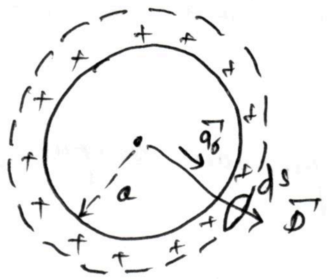
$$2D = \rho_s$$

$$\boxed{D = \frac{\rho_s}{2}}$$

$$\therefore \vec{D} = \frac{\rho_s}{2}\,\vec{a_z} \quad C/m^2$$

$$\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{\rho_s}{2\epsilon}\,\vec{a_z} \quad V/m.$$

3. SPHERICAL SHELL OR THE SPHERE:



— Consider a Sphere of radius a with charge density ρ_s C/m^2.

Case(i) $(r > a)$

$$Q = \int \vec{D} \cdot \vec{ds}$$

The direction of $\vec{D}$ is in $\vec{a_r}$ direction

$$\vec{D} = D\,\vec{a_r} \quad \text{———} \quad (1)$$

$$\vec{ds} = r^2 \sin\theta \, d\theta \, d\phi \, \vec{a_r}$$

$$\vec{D} \cdot \vec{ds} = D r^2 \sin\theta \, d\theta \, d\phi$$

$$Q = \int_S D r^2 \sin\theta \, d\theta \, d\phi = D r^2 \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$Q = D r^2 \int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi} d\phi$$

$$= 2 D r^2 \left[\phi\right]_0^{2\pi}$$

$$Q = 4\pi r^2 D$$

$$D = \frac{Q}{4\pi r^2}$$

$$\therefore \vec{D} = \frac{Q}{4\pi r^2}\,\vec{a_r} \quad C/m^2$$

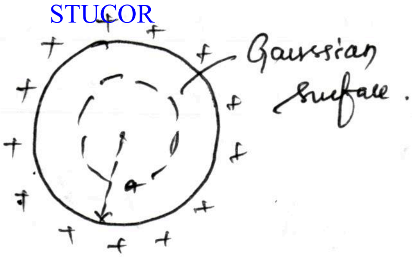
$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\,\vec{a_r} \quad V/m.$$

Case(ii): $(r = a)$

$$\vec{D} = \frac{Q}{4\pi a^2}\,\vec{a_r} \quad C/m^2$$

$$\vec{E} = \frac{Q}{4\pi\epsilon a^2}\,\vec{a_r} \quad V/m$$

 Case (iii): $(r < a)$



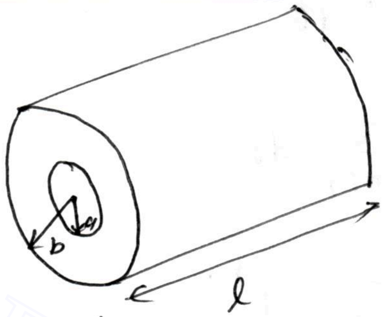
Here no charge is enclosed by a spherical shell. So total charge enclosed is zero.

$$Q = 0$$

$$\vec{D} = 0$$

$$\vec{E} = 0.$$

4. $\vec{D}$ DUE TO COAXIAL CYLINDER:



By Gauss law,

$$\psi = Q$$

$$\psi = \rho_L l$$

$$\int \vec{D} \cdot \vec{ds} = \rho_L l$$

$$D \times 2\pi r \cancel{l} = \rho_L \cancel{l}$$

$$D = \frac{\rho_L}{2\pi r}$$

The direction of $\vec{D}$ is in $\vec{a_r}$ direction

So $\vec{D} = D \vec{a_r}$

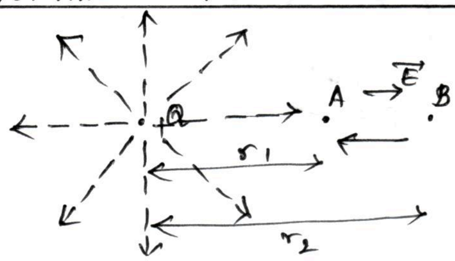
$$\boxed{\vec{D} = \frac{\rho_L}{2\pi r} \vec{a_r}}$$

$$\vec{E} = \frac{\vec{D}}{\epsilon}$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi \epsilon r} \vec{a_r}}$$

POTENTIAL DIFFERENCES FOR DIFFERENT CONFIGURATIONS:

1. POTENTIAL DIFFERENCE FOR A POINT CHARGE:



The electric field intensity of a point charge

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\vec{a_r}$$

$$\vec{dr} = dr\,\vec{a_r}$$

$$\vec{E}\cdot\vec{dr} = \frac{Q}{4\pi\epsilon r^2}\,dr$$

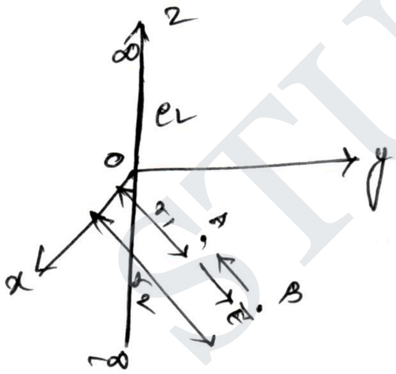
Potential, $V = -\int \vec{E}\cdot d\vec{l}$

$$= -\int_{r_2}^{r_1} \frac{Q}{4\pi\epsilon r^2}\,dr$$

$$= \frac{-Q}{4\pi\epsilon}\left[\frac{-1}{r}\right]_{r_2}^{r_1}$$

$$\boxed{V = \frac{Q}{4\pi\epsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \text{ Volts.}$$

2. POTENTIAL DUE TO INFINITE LINE CHARGE:



The $\vec{E}$ for infinite line charge is

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon\rho}\vec{a_\rho}$$

$$\vec{dr} = d\rho\,\vec{a_\rho}$$

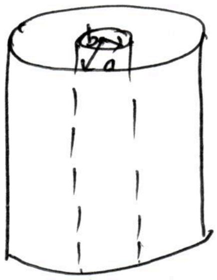
$$\vec{E}\cdot\vec{dr} = \frac{\rho_L}{2\pi\epsilon\rho}\,d\rho$$

$$V = -\int_{r_2}^{r_1} \vec{E}\cdot\vec{dl} = -\int_{r_2}^{r_1}\frac{\rho_L}{2\pi\epsilon\rho}\,d\rho$$

$$= \frac{-\rho_L}{2\pi\epsilon}\left[\ln\rho\right]_{r_2}^{r_1} = \frac{-\rho_L}{2\pi\epsilon}\ln\left(\frac{r_1}{r_2}\right)$$

$$\boxed{V = \frac{\rho_L}{2\pi\epsilon}\ln\left(\frac{r_2}{r_1}\right) \text{ Volts}}$$

3. POTENTIAL DUE TO COAXIAL CYLINDER :



The $\vec{E}$ for a line charge is

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon\rho}\vec{a_\rho}$$

$$\vec{dr} = d\rho\,\vec{a_\rho}$$

$$\vec{E}\cdot\vec{dr} = \frac{\rho_L}{2\pi\epsilon\rho}d\rho.$$

$$V = -\int_b^a \vec{E}\cdot\vec{dr} = -\int_b^a \frac{\rho_L}{2\pi\epsilon\rho}d\rho$$

$$= \frac{-\rho_L}{2\pi\epsilon}\left[\ln\rho\right]_b^a$$

$$= \frac{-\rho_L}{2\pi\epsilon}\ln\left(\frac{a}{b}\right)$$

$$\boxed{V = \frac{\rho_L}{2\pi\epsilon}\ln\left(\frac{b}{a}\right)} \text{ Volts.}$$

4. POTENTIAL DUE TO SPHERE:

Case(i) $(r > a)$

$$V = -\int_\infty^r \vec{E}\cdot\vec{dr} = -\int_\infty^r \frac{Q}{4\pi\epsilon r^2}dr = \frac{-Q}{4\pi\epsilon}\left[\frac{-1}{r}\right]_\infty^r$$

$$\boxed{V = \frac{Q}{4\pi\epsilon r} \text{ Volts}}$$

Case(ii) $(r = a)$

$$V = -\int_\infty^a \vec{E}\cdot\vec{dr} = -\int_\infty^a \frac{Q}{4\pi\epsilon r^2}dr = \frac{-Q}{4\pi\epsilon}\left[\frac{-1}{r}\right]_\infty^a$$

$$\boxed{V = \frac{Q}{4\pi\epsilon a} \text{ Volts}}$$

Case(iii) $(r < a)$

when $r < a$, the electric field intensity $(\vec{E}) = 0$.

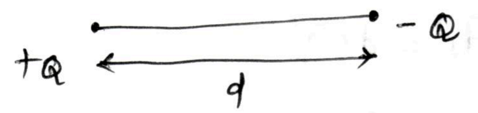
(i.e) it requires no work to move a test charge inside the shell.

Hence the electric charge inside the shell is constant.

$$\boxed{V = \frac{Q}{4\pi\epsilon a} \text{ Volts}}$$

ELECTRIC DIPOLE:

Two equal and opposite charges are separated by a small distance is called electric dipole.



DIPOLE MOMENT:

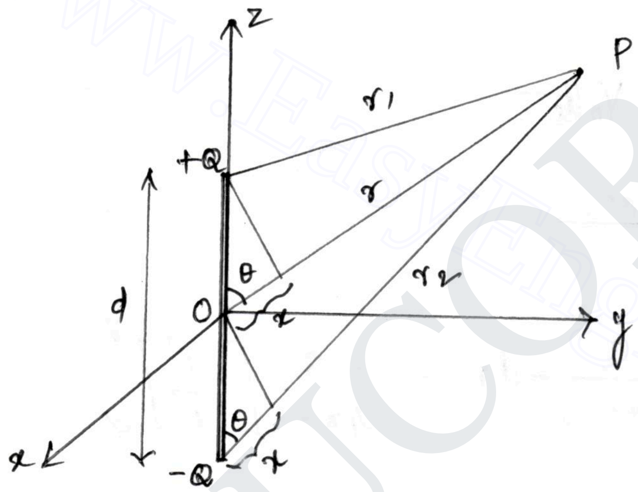
The Product of charge and distance between the charges is called dipole moment

$$p = Qd$$

POLARIZATION:

Dipole moment per unit volume is called polarization.

ELECTRIC FIELD AND POTENTIAL DUE TO DIPOLE:



– Let P be any point at a distance of r_1, r_2 & r from +Q, -Q and the midpoint of the dipole respectively.

Potential at point P due to +Q is

$$V_1 = \frac{Q}{4\pi\epsilon r_1}$$

Potential at point P due to -Q is

$$V_2 = \frac{-Q}{4\pi\epsilon r_2}$$

The Potential at point P due to the dipole is

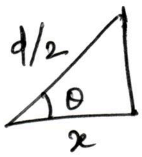
$$V = V_1 + V_2$$

$$V = \frac{Q}{4\pi\epsilon r_1} - \frac{Q}{4\pi\epsilon r_2}$$

$$V = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

from the figure,

$$r_1 = r - x$$
$$r_2 = r + x$$



$$\cos\theta = \frac{x}{d/2} \Rightarrow x = (d/2)\cos\theta$$

$$\therefore r_1 = r - \frac{d}{2}\cos\theta$$
$$r_2 = r + \frac{d}{2}\cos\theta$$

$$\Rightarrow V = \frac{Q}{4\pi\epsilon}\left[\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right]$$

$$V = \frac{Q}{4\pi\epsilon}\left[\frac{r + \frac{d}{2}\cos\theta - r + \frac{d}{2}\cos\theta}{r^2 - \frac{d^2}{4}\cos^2\theta}\right]$$

$$V = \frac{Q\,d\cos\theta}{4\pi\epsilon\left(r^2 - \frac{d^2}{4}\cos^2\theta\right)}$$

$$r >> \frac{d}{2}$$

$$\boxed{V = \frac{Q\,d\cos\theta}{4\pi\epsilon r^2}}$$

(or)

$$\boxed{V = \frac{p\cos\theta}{4\pi\epsilon r^2}} \qquad \because p = Qd.$$

$$\boxed{V = \frac{\bar{p}\cdot\bar{a}_r}{4\pi\epsilon r^2}} \qquad \because \bar{p}\cdot\bar{a}_r = p\cos\theta.$$

The electric field of a dipole is

$$\vec{E} = -\nabla V$$

$$= -\left[\frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}\right]$$

$$\frac{\partial V}{\partial r} = \frac{-2Qd\cos\theta}{4\pi\epsilon r^3}$$

$$\frac{\partial V}{\partial \theta} = \frac{-Qd\sin\theta}{4\pi\epsilon r^2}$$

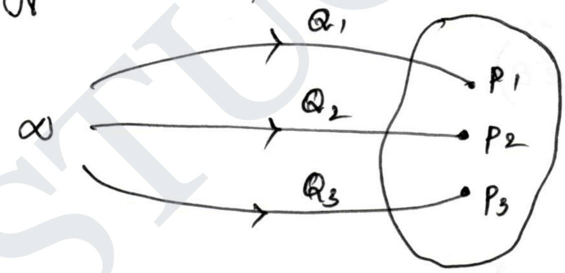
$$\frac{\partial V}{\partial \phi} = 0.$$

$$\vec{E} = \frac{2Qd\cos\theta}{4\pi\epsilon r^3}\vec{a_r} + \frac{Qd\sin\theta}{4\pi\epsilon r^3}\vec{a_\theta}$$

$$\boxed{\vec{E} = \frac{Qd}{4\pi\epsilon r^3}\left[2\cos\theta\,\vec{a_r} + \sin\theta\,\vec{a_\theta}\right] \quad V/m}$$

ELECTROSTATIC ENERGY AND ENERGY DENSITY:

- Energy stored per unit volume as the volume tends to zero.



- Three point charges Q_1, Q_2 and Q_3 are positioned in free space.
- No work is required to transfer Q_1 from ∞ to P_1 because the space is initially charge free. $(W_1 = 0)$
- The workdone in transferring Q_2 from ∞ to P_2 is equal to the product of Q_2 with potential V_{21} at point P_2 due to Q_1.

$$(W_2 = Q_2 V_{21})$$

- The workdone in transferring Q_3 from ∞ to P_3 is equal to

$$W_3 = Q_3(V_{32} + V_{31}).$$

The total workdone in positioning the charge $\Big\}$ $W = W_1 + W_2 + W_3$

$$= Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \quad \text{——} (1)$$

If the same charges are positioned in reverse order,

The Total workdone in positioning the charge $\Big\}$ $W = W_3 + W_2 + W_1$

$$= Q_2 V_{23} + Q_1 (V_{12} + V_{13}) \quad \text{——} (2)$$

$(1) + (2) \Rightarrow$

$$2W = Q_1 (V_{12} + V_{13}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$$

$$2W = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

$$W = \frac{1}{2} (Q_1 V_1 + Q_2 V_2 + Q_3 V_3) \quad \text{——} (3)$$

If there are n point charges

$$W = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k$$

Instead of point charges, the region has continuous charge distribution.

$$W = \frac{1}{2} \int_L \rho_L V \, dl \quad \text{for line charge} \quad \text{——} (4)$$

$$W = \frac{1}{2} \int_S \rho_S V \, ds \quad \text{for surface charge} \quad \text{——} (5)$$

$$W = \frac{1}{2} \int_V \rho_v V \, dv \quad \text{for volume charge.} \quad \text{——} (6)$$

Sub $\rho_v = \nabla \cdot \vec{D}$ in eq (6)

$$W = \frac{1}{2} \int_V (\nabla \cdot \vec{D}) V \, dv$$

$$= \frac{1}{2} \int_V [\nabla \cdot (V \vec{D}) - \vec{D} \cdot (\nabla V)] \, dv$$

$$= \frac{1}{2} \int_V \nabla \cdot (V \vec{D}) \, dv - \frac{1}{2} \int \vec{D} \cdot (\nabla V) \, dv$$

By divergence theorem 1^{st} term will be

$$W = \frac{1}{2}\int_S V\vec{D}\cdot d\vec{s} - \frac{1}{2}\int \vec{D}\cdot \nabla V\, dv$$

$\Downarrow 0$

If S is large.

$$W = -\frac{1}{2}\int \vec{D}\cdot(\nabla V)\, dv$$

Sub $\nabla V = -\vec{E}$

$$W = -\frac{1}{2}\int \vec{D}\cdot(-\vec{E})\, dv$$

$$\boxed{W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, dv} \quad \text{——} \quad (7)$$

Sub $\vec{D} = \epsilon\vec{E}$ in eqn (7)

$$W = \frac{1}{2}\int \epsilon\vec{E}\cdot\vec{E}\, dv$$

$$\boxed{W = \frac{1}{2}\int \epsilon E^2\, dv} \quad \text{——} \quad (8)$$

Sub $\vec{E} = \frac{\vec{D}}{\epsilon}$ in eqn (7)

$$W = \frac{1}{2}\int \vec{D}\cdot\frac{\vec{D}}{\epsilon}\, dv$$

$$\boxed{W = \frac{1}{2}\int \frac{D^2}{\epsilon}\, dv}.$$

GAUSS DIVERGENCE THEOREM:

The total outward flux of vector field $\vec{D}$ through the closed surface S is same as the volume integral of the divergence of $\vec{D}$.

$$\int_S \vec{D}\cdot d\vec{s} = \int_V (\nabla\cdot\vec{D})\, dv.$$

POISSON'S AND LAPLACE'S EQUATION:

The point form of Gauss law is

$$\nabla \cdot \vec{D} = \rho_v$$

Sub $\vec{D} = \epsilon \vec{E}$

$$\nabla \cdot (\epsilon \vec{E}) = \rho_v$$

$$\epsilon (\nabla \cdot \vec{E}) = \rho_v$$

Sub $E = -\nabla V$

$$\epsilon [\nabla \cdot (-\nabla V)] = \rho_v$$

$$-\epsilon [\nabla \cdot \nabla V] = \rho_v$$

$$-\epsilon [\nabla^2 V] = \rho_v$$

$$\boxed{\nabla^2 V = \frac{-\rho_v}{\epsilon}} \Rightarrow \text{Poisson's equation.}$$

In charge free medium, the volume charge density is zero

$$\boxed{\nabla^2 V = 0} \Rightarrow \text{Laplace's equation.}$$

Poisson's equation in Cartesian:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho_v}{\epsilon}$$

Poisson's equation in Cylindrical:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho_v}{\epsilon}$$

Poisson's equation in Spherical:

$$\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial \theta} \sin\theta\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2} = \frac{-\rho_v}{\epsilon}$$

Laplace's equation in Cartesian:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace's equation in Cylindrical:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Laplace's equation in Spherical:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0.$$

PROBLEM:

① Determine whether the following potentials satisfy the Laplace's equation or not.

(i) $V = x^2 - y^2 + z^2$

(ii) $V = \rho\cos\phi + z$

(iii) $V = r\cos\theta + \phi$

Solution:

for Cartesian:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}$$

$$=\frac{\partial^2}{\partial x^2}(x^2-y^2+z^2)+\frac{\partial^2}{\partial y^2}(x^2-y^2+z^2)+\frac{\partial^2}{\partial z^2}(x^2-y^2+z^2)$$

$$= 2 - 2 + 2$$

$$\boxed{\nabla^2 V = 2} \neq 0$$

∴ The given potential doesn't satisfy the Laplace equation.

for Cylindrical:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial^2 V}{\partial z^2}$$

$$=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}(\rho\cos\phi+z)\right]+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}(\rho\cos\phi+z)+\frac{\partial^2}{\partial z^2}(\rho\cos\phi+z)$$

$$=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cos\phi)+\frac{1}{\rho^2}(-\rho\cos\phi)+0$$

$$=\frac{1}{\rho}(\cos\phi)-\frac{1}{\rho}(\cos\phi)$$

$$\boxed{\nabla^2 V = 0}$$

∴ The given Potential Satisfies the Laplace equation.

for Spherical:

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}.$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}(r\cos\theta+\phi)\right]+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(r\cos\theta+\phi)\right]+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}(r\cos\theta+\phi)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\cos\theta) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(-r\sin^2\theta) + \frac{1}{r^2\sin^2\theta}(0)$$

$$= \left(\frac{1}{r^2}\times 2r\cos\theta\right) + \frac{(-r)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{2\cos\theta}{r} - \frac{1}{r\sin\theta}\left(\frac{\cancel{2}\sin 2\theta}{\cancel{2}}\right)$$

$$= \frac{2\cos\theta}{r} - \frac{2\cancel{\sin\theta}\cos\theta}{r\cancel{\sin\theta}}$$

$$= \frac{2\cos\theta}{r} - \frac{2\cos\theta}{r}$$

$$\boxed{\nabla^2 V = 0}$$

$\therefore$ Given potential satisfies the Laplace's equation.

CAPACITANCE:

Capacitance between two Conductors is defined as the ratio of the magnitude of the total charge on either Conductor to the potential difference between the Conductors.

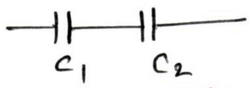
$$\boxed{C = \frac{Q}{V}}$$

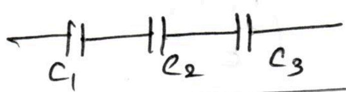
$$Q = \int \vec{D}\cdot\vec{ds} = \int \epsilon\vec{E}\cdot\vec{ds}$$

$$V = -\int \vec{E}\cdot\vec{dr}$$

$$C = \frac{\epsilon\int \vec{E}\cdot\vec{ds}}{-\int \vec{E}\cdot\vec{dr}}$$

CAPACITORS IN SERIES:

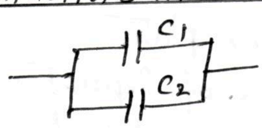


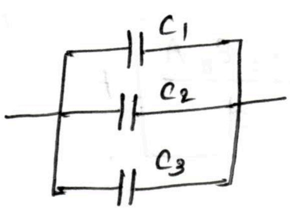


$$\boxed{C_{eq} = \frac{C_1 C_2}{C_1 + C_2}}$$

$$\boxed{C_{eq} = \frac{C_1 C_2 C_3}{C_1C_2 + C_2C_3 + C_3C_1}}$$

CAPACITORS IN PARALLEL:

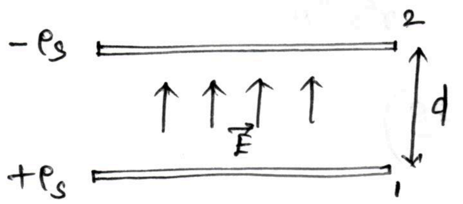




$$\boxed{C_{eq} = C_1 + C_2 + C_3}$$

$$\boxed{C_{eq} = C_1 + C_2}$$

PARALLEL PLATE CAPACITOR



– Consider two parallel plates having charge density $+\rho_s$ & $-\rho_s$ C/m² separated by a distance d.

The electric field at plate 1 } $\vec{E_1} = \frac{\rho_s}{2\varepsilon}\vec{a_z}$

The electric field at plate 2 } $\vec{E_2} = \frac{-\rho_s}{2\varepsilon}(-\vec{a_z}) = \frac{\rho_s}{2\varepsilon}\vec{a_z}$

$$\vec{E} = \vec{E_1} + \vec{E_2} = \frac{\rho_s}{\varepsilon}\vec{a_z}$$

$$\vec{dr} = dz\,\vec{a_z}$$

$$\vec{E}\cdot\vec{dr} = \frac{\rho_s}{\varepsilon}dz$$

The Potential difference between the plates } $V = -\int_{z=d}^{0} \vec{E}\cdot\vec{dr}$

$$= -\int_{d}^{0} \frac{\rho_s}{\varepsilon}dz$$

$$= \frac{-\rho_s}{\varepsilon}\left[z\right]_d^0$$

$$V = \frac{\rho_s}{\varepsilon}d \quad \text{——— (1)}$$

Capacitance, $C = \frac{Q}{V}$ ——— (2)

for surface charge, $Q = \rho_s A$ ——— (3)

Sub (1) & (3) in (2)

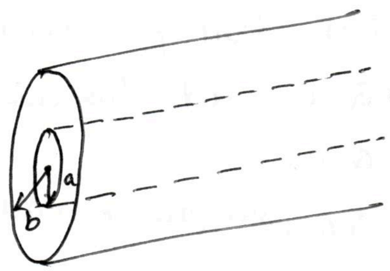
$$C = \frac{\rho_s A}{\left(\frac{\rho_s d}{\varepsilon}\right)}$$

$$C = \frac{\varepsilon A}{d}$$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r A}{d}}\ F$$

where
A → Area of plates
d → distance of Separation
ε_r → Relative permittivity.

Coaxial Capacitor :



– Consider a Coaxial Cable having inner radius 'a' m with line charge density of ρ_L C/m and the outer radius 'b' m with line charge density $-\rho_L$ C/m.

The Electric field for a line charge,

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon\rho} \vec{a_\rho}$$

$$\vec{dr} = d\rho \, \vec{a_\rho}$$

$$\vec{E} \cdot \vec{dr} = \frac{\rho_L}{2\pi\epsilon\rho} d\rho.$$

Potential difference, $V = -\int \vec{E} \cdot \vec{dr}$

$$= -\int_b^a \frac{\rho_L}{2\pi\epsilon\rho} d\rho$$

$$= \frac{-\rho_L}{2\pi\epsilon} \left[\ln \rho \right]_b^a$$

$$= \frac{-\rho_L}{2\pi\epsilon} \ln\left(\frac{a}{b}\right)$$

$$V = \frac{\rho_L}{2\pi\epsilon} \ln\left(\frac{b}{a}\right) \quad \text{———} \quad (1)$$

for line charge, $Q = \rho_L L$ ——— (2)

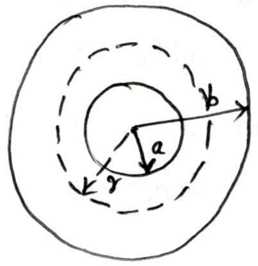
Capacitance, $C = \frac{Q}{V}$

Sub (1) & (2)

$$C = \frac{\rho_L L}{\frac{\rho_L}{2\pi\epsilon} \ln\left(\frac{b}{a}\right)}$$

$$\boxed{C = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}} \quad F$$

SPHERICAL CAPACITOR:



- Consider a Spherical shell of inner radius 'a' m with a charge +Q C and the outer radius 'b' m with a charge −Q C.
- Consider a gaussian surface of radius r.

The Electric field between two shells is

$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\vec{u_r} \qquad a \le r \le b.$$

$$\vec{dr} = dr\,\vec{u_r}$$

$$\vec{E}\cdot\vec{dr} = \frac{Q}{4\pi\epsilon r^2}dr$$

The potential rise from b to a is $\Big\}$ $V = -\int_b^a \vec{E}\cdot\vec{dr}$

$$V = -\int_b^a \frac{Q}{4\pi\epsilon r^2}dr$$

$$= \frac{-Q}{4\pi\epsilon}\left[\frac{-1}{r}\right]_b^a$$

$$= \frac{-Q}{4\pi\epsilon}\left[\frac{-1}{a} + \frac{1}{b}\right]$$

$$V = \frac{Q}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right) \quad \text{———} \quad (1)$$

Capacitance, $C = \frac{Q}{V}$

sub eqn (1)

$$C = \frac{\cancel{Q}}{\frac{\cancel{Q}}{4\pi\epsilon}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$\boxed{C = \frac{4\pi\epsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)}} \text{ F}$$

CAPACITANCE OF SINGLE ISOLATED SPHERE:

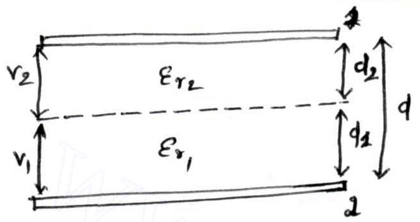
If outer sphere is infinitely large (i,e) $b = \infty$

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{\infty}}$$

$$= \frac{4\pi\varepsilon}{(1/a)}$$

$$\boxed{C = 4\pi\varepsilon a}$$

PARALLEL PLATE CAPACITOR WITH TWO DIFFERENT DIELECTRICS:



- Consider a parallel plate capacitor with two different dielectrics separated by a distance d.

The total potential due to two plates is

$$V = V_1 + V_2 \quad \text{——— (1)}$$

here $V_1 = E_1 d_1$

$V_2 = E_2 d_2$

$$\therefore (1) \Rightarrow V = E_1 d_1 + E_2 d_2$$

$$= \frac{D_1}{\varepsilon_1} d_1 + \frac{D_2}{\varepsilon_2} d_2$$

$$V = \frac{D}{\varepsilon_1} d_1 + \frac{D}{\varepsilon_2} d_2$$

$\because$ for dielectric $D_1 = D_2 = D$

The flux density D can be written as, $D = \frac{Q}{A}$

$$V = \frac{Q}{\varepsilon_1 A} d_1 + \frac{Q}{\varepsilon_2 A} d_2$$

$$= Q\left[\frac{d_1}{\varepsilon_1 A} + \frac{d_2}{\varepsilon_2 A}\right]$$

$$\frac{Q}{V} = \frac{1}{\frac{d_1}{\varepsilon_1 A} + \frac{d_2}{\varepsilon_2 A}} = \frac{1}{\frac{1}{\left(\frac{\varepsilon_1 A}{d_1}\right)} + \frac{1}{\left(\frac{\varepsilon_2 A}{d_2}\right)}}$$

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

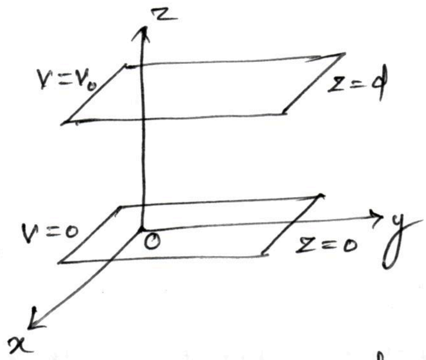
where $C_1 = \frac{\varepsilon_1 A}{d_1}$

$C_2 = \frac{\varepsilon_2 A}{d_2}$

CAPACITANCE USING LAPLACE EQUATIONS:

CAPACITANCE BETWEEN PARALLEL PLATES USING LAPLACE EQUATION:

— Consider two parallel plates placed parallel to xy plane. They are separated by a distance 'd' and this space is filled with dielectric having permittivity ϵ.



— The lower plate is maintained at potential zero ($V=0$) and upper plate at ($V=V_0$). So V depends only on z.

The Laplace equation is

$$\nabla^2 V = 0$$

for the component z

$$\frac{\partial^2 V}{\partial z^2} = 0 \quad \text{——— (1)}$$

Integrating equation (1)

$$\frac{\partial V}{\partial z} = C_1 \quad \text{——— (2)}$$

Integrating equation (2)

$$V = C_1 z + C_2 \quad \text{——— (3)}$$

at $z=0$, $V=0$

$$(3) \Rightarrow 0 = C_1(0) + C_2$$

$$\boxed{C_2 = 0}$$

Sub $C_2 = 0$ in equation (3)

$$V = C_1 z \quad \text{——— (4)}$$

at $z=d$, $V=V_0$

$$(4) \Rightarrow V_0 = C_1 d$$

$$\boxed{C_1 = \frac{V_0}{d}}$$

Sub C_1 value in equation (4)

$$\boxed{V = \frac{V_0}{d} z}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)=0$$

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)=0 \quad \text{--- (1)}$$

Integrating equation (1)

$$\rho\frac{\partial V}{\partial\rho}=c_1$$

$$\frac{\partial V}{\partial\rho}=\frac{c_1}{\rho} \quad \text{--- (2)}$$

Integrating equation (2)

$$\boxed{V=c_1\ln\rho+c_2} \quad \text{--- (3)}$$

at $\rho=b$, $V=0$

$$(3) \Rightarrow \quad 0=c_1\ln b+c_2 \quad \text{--- (4)}$$

at $\rho=a$, $V=V_0$

$$(3) \Rightarrow \quad V_0=c_1\ln a+c_2 \quad \text{--- (5)}$$

$$(4)-(5) \Rightarrow \quad -V_0=c_1\ln b-c_1\ln a$$

$$=c_1\ln\left(\frac{b}{a}\right)$$

$$\boxed{c_1=\frac{-V_0}{\ln\left(\frac{b}{a}\right)}}$$

from equation (4)

$$c_2=-c_1\ln b$$

sub c_1 value

$$\boxed{c_2=\frac{V_0\ln b}{\ln\left(\frac{b}{a}\right)}}$$

$$\therefore V=\left[\frac{-V_0}{\ln\left(\frac{b}{a}\right)}\right]\ln\rho+\left[\frac{V_0\ln b}{\ln\left(\frac{b}{a}\right)}\right]$$

The relation between $\vec{E}$ and V is

$$\vec{E}=-\nabla V$$

for the Component ρ

 The relation between $\vec{E}$ and V is

$$\vec{E} = -\nabla V$$

for the Component z

$$\vec{E} = -\frac{\partial V}{\partial z}\vec{a_z}$$

$$\boxed{\vec{E} = \frac{-V_0}{d}\vec{a_z}}$$

$$\vec{D} = \varepsilon\vec{E}$$

$$\vec{D} = -\frac{\varepsilon V_0}{d}\vec{a_z}$$

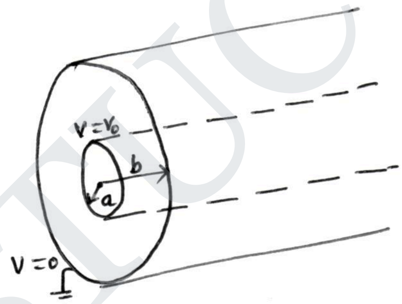
Charge density, $\rho_s = |\vec{D}| = \frac{\varepsilon V_0}{d}$

Total charge, $Q = \rho_s A = \frac{\varepsilon V_0 A}{d}$

$$\frac{Q}{V_0} = \frac{\varepsilon A}{d}$$

$$\boxed{C = \frac{\varepsilon A}{d}} \quad F.$$

CAPACITANCE BETWEEN COAXIAL CYLINDERS USING LAPLACE EQUATION :



- Consider a Coaxial cylinder of length L with inner radius 'a' and outer radius 'b'. At $r=a$, the potential $V=V_0$ and at $r=b$, the potential is zero.

- The electric field intensity is in radial direction from inner to outer Cylinder, hence V is a function of ρ only and not the function of ϕ and z.

By Laplace equation

$$\nabla^2 V = 0$$

for the Component ρ

$$\vec{E} = -\frac{\partial V}{\partial \rho}\vec{a_\rho}$$

$$= \frac{V_0}{\rho \ln(b/a)}\vec{a_\rho}$$

$$\vec{D} = \epsilon\vec{E}$$

$$\vec{D} = \frac{\epsilon V_0}{\rho \ln(b/a)}\vec{a_\rho}$$

Charge density, $\rho_s = |\vec{D}| = \dfrac{\epsilon V_0}{\rho \ln(b/a)}$

Total charge, $Q = \rho_s \times$ Area of Cylinder

$$Q = \rho_s \times 2\pi\rho L$$

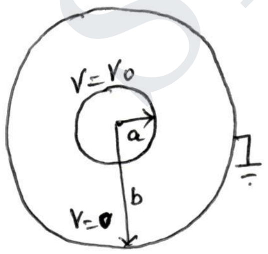
Sub $\rho_s = \dfrac{\epsilon V_0}{\rho \ln(\frac{b}{a})}$

$$Q = \frac{\epsilon V_0}{\cancel{\rho} \ln(\frac{b}{a})} \times 2\pi\cancel{\rho} L$$

$$\frac{Q}{V_0} = \frac{2\pi\epsilon L}{\ln(\frac{b}{a})}$$

$$\boxed{C = \frac{2\pi\epsilon L}{\ln(\frac{b}{a})}} \quad F$$

CAPACITANCE BETWEEN SPHERICAL SHELLS USING LAPLACE EQUATION:



– Consider two spherical conducting shells separated by dielectric with permittivity ϵ.

– The radius of inner shell is 'a' and it has a potential as V_0

– The radius of outer shell is 'b' and it has potential as zero.

– The electric field intensity is in radial direction from inner shell to outer shell, hence V is a function of r only, not as the the function of θ and ϕ.

By Laplace's equation

$$\nabla^2 V = 0$$

for the Component r,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0 \quad \text{—— (1)}$$

Integrating equation (1)

$$r^2\frac{\partial V}{\partial r} = c_1$$

$$\frac{\partial V}{\partial r} = \frac{c_1}{r^2} \quad \text{—— (2)}$$

Integrating equation (2)

$$V = \frac{-c_1}{r} + c_2 \quad \text{—— (3)}$$

at $r = b$, $V = 0$

$$0 = \frac{-c_1}{b} + c_2 \quad \text{—— (4)}$$

at $r = a$, $V = V_0$

$$V_0 = \frac{-c_1}{a} + c_2 \quad \text{—— (5)}$$

(4) – (5) $\Rightarrow$

$$-V_0 = \frac{-c_1}{b} + \frac{c_1}{a}$$

$$-V_0 = c_1\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\boxed{c_1 = \frac{-V_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$$

from equation (4)

$$c_2 = \frac{c_1}{b}$$

$$\boxed{c_2 = \frac{-V_0}{b\left(\frac{1}{a} - \frac{1}{b}\right)}}$$

$$\therefore V = \frac{-V_0}{r\left(\frac{1}{a} - \frac{1}{b}\right)} - \frac{V_0}{b\left(\frac{1}{a} - \frac{1}{b}\right)} \quad \text{—— (6)}$$

The relation between $\vec{E}$ and V is

$$\vec{E} = -\nabla V$$

for the Component r

$$\vec{E} = -\frac{\partial V}{\partial r}\vec{a}_r$$

$$\vec{E} = \frac{V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} \vec{a_r}$$

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{D} = \frac{\epsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} \vec{a_r}$$

The charge density, $\rho_s = |\vec{D}| = \dfrac{\epsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}$

Total charge, $Q = \rho_s \times$ Area of a sphere

$$Q = \rho_s \times 4\pi r^2$$

Sub $\rho_s = \dfrac{\epsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}$

$$Q = \frac{\epsilon V_0}{\cancel{r^2}\left(\frac{1}{a}-\frac{1}{b}\right)} \times 4\pi \cancel{r^2}$$

$$\frac{Q}{V_0} = \frac{4\pi\epsilon}{\frac{1}{a}-\frac{1}{b}}$$

$$\boxed{C = \frac{4\pi\epsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)}} \; F.$$

BOUNDARY CONDITIONS FOR ELECTRIC FIELDS:

– The conditions existing at the boundary of the two medium when the field passes from one medium to other are called as boundary conditions.

– To analyze the boundary conditions the following equations are required.

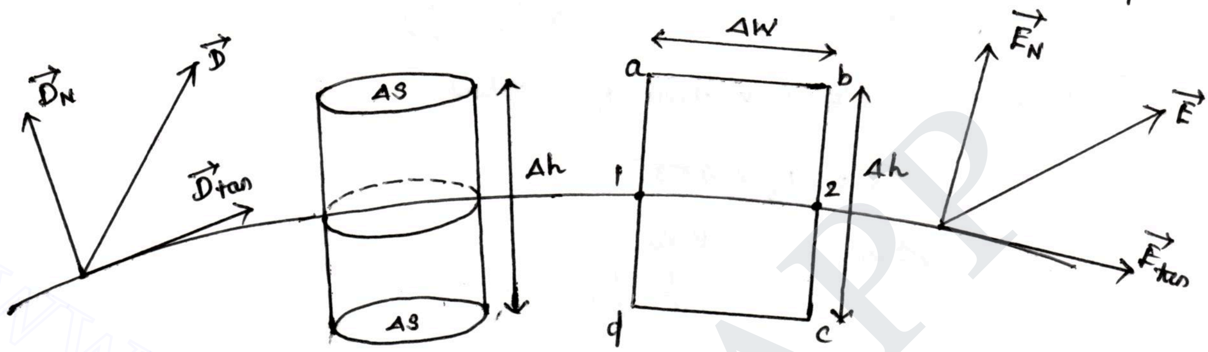
(i) $\oint \vec{E} \cdot \vec{dl} = 0$

(ii) $\oint \vec{D} \cdot \vec{ds} = Q$

– The field intensity $(\vec{E})$ and flux density $(\vec{D})$ is required to be decomposed into tangential component and normal component.

BOUNDARY CONDITIONS BETWEEN CONDUCTOR AND FREE SPACE :

- Consider a boundary between Conductor and free space.
- The Conductor is ideal having infinite Conductivity. Thus $\vec{E}$, $\vec{D}$ and ρ_v within the Conductor are zero. ρ_s is the surface charge density on the surface of the Conductor.
- To determine the boundary Conditions the closed path (abcda) and the gaussian surface as cylinder are used.



Tangential Components at the boundary :

The workdone in carrying unit positive charge along a closed path is zero. (i,e)

$$\int \vec{E} \cdot \vec{dl} = 0 \quad \text{——— (1)}$$

For a closed path abcda equation (1) can be written as, sum of four parts

$$\int_a^b \vec{E} \cdot \vec{dl} + \int_b^c \vec{E} \cdot \vec{dl} + \int_c^d \vec{E} \cdot \vec{dl} + \int_d^a \vec{E} \cdot \vec{dl} = 0 \quad \text{——— (2)}$$

The rectangular closed path has a height Δh and width Δw. The rectangular closed path is placed in such a way that half of it is in the conductor and remaining half is in free space.

The portion c-d is in the Conductor medium where $\vec{E} = 0$.

$$\therefore \int_c^d \vec{E} \cdot \vec{dl} = 0 \quad \text{——— (A)}$$

As Δw is very small, $\vec{E}$ over it is assumed as Constant.

(i,e) $\vec{E} = E_{tan}$ ($\because$ Δw is along tangential direction).

$$\int_a^b \vec{E} \cdot \vec{dl} = E_{tan} \int_a^b \vec{dl} = E_{tan} (\Delta w) \quad \text{——— (B)}$$

 As Δh is very small, $\vec{E}$ over it is assumed as Constant.

(i.e) $\vec{E} = E_N$ $\quad$ ($\because \Delta h$ is along normal direction)

$$\int_b^c \vec{E}\cdot\vec{dl} = \int_b^2 \vec{E}\cdot\vec{dl} + \int_2^c \vec{E}\cdot\vec{dl}.$$

$$= E_N \int_b^2 \vec{dl} + 0 \qquad (\because 2-c \text{ lies in Conductor})$$

$$= E_N\left(\frac{\Delta h}{2}\right) \text{ ——— } (C)$$

Similarly for the path d-a, the Condition is same as for the path b-c, only the direction is opposite.

$$\int_d^a \vec{E}\cdot\vec{dl} = -E_N\left(\frac{\Delta h}{2}\right) \text{ ——— } (D)$$

Sub equations (A), (B), (C) & (D) in equation (2)

$$(2) \Rightarrow E_{tan}(\Delta w) + \cancel{E_N\left(\frac{\Delta h}{2}\right)} + 0 - \cancel{E_N\left(\frac{\Delta h}{2}\right)} = 0$$

$$E_{tan}\,\Delta w = 0$$

$$\boxed{E_{tan} = 0} \text{ ——— } (3)$$

WKT $\quad D_{tan} = \varepsilon E_{tan}$

$$= \varepsilon \times 0$$

$$\boxed{D_{tan} = 0} \text{ ——— } (4)$$

The tangential Components of electric field intensity and electric flux density are zero at the boundary between Conductor and freespace.

<u>Normal Components at the boundary</u>:

By Gauss's law

$$\int \vec{D}\cdot\vec{ds} = Q \text{ ——— } (5)$$

For the gaussian surface equation (5) Can be written as

$$\int_{top} \vec{D}\cdot\vec{ds} + \int_{bottom} \vec{D}\cdot\vec{ds} + \int_{lateral} \vec{D}\cdot\vec{ds} = Q \text{ ——— } (6)$$

 The bottom surface is in the conductor where $\vec{D'} = 0$

$$\therefore \int_{bottom} \vec{D'} \cdot \vec{ds} = 0 \quad \text{——— (E)}$$

For the lateral surface, the area is $2\pi r(\Delta h)$.

As Δh tends to zero, area of lateral surface is zero.

$$\therefore \int_{lateral} \vec{D'} \cdot \vec{ds'} = 0 \quad \text{——— (F)}$$

For smaller top surface $\vec{D'}$ is assumed as constant.

(i,e) $\vec{D'} = D_N$

$$\therefore \int_{top} \vec{D'} \cdot \vec{ds'} = D_N \int_{top} \vec{ds'} = D_N \Delta S \quad \text{——— (G)}$$

Sub equations (E), (F) and (G) in equation (6)

$$D_N \Delta S = Q$$

$$D_N \cancel{\Delta S} = \rho_S \cancel{\Delta S}$$

$$\boxed{D_N = \rho_S} \quad \text{——— (7)}$$

WKT

$$D_N = \varepsilon E_N$$

$$E_N = \frac{D_N}{\varepsilon}$$

Sub $D_N = \rho_s$ and $\varepsilon = \varepsilon_0 \varepsilon_r$

$$= \frac{\rho_S}{\varepsilon_0 \varepsilon_r}$$

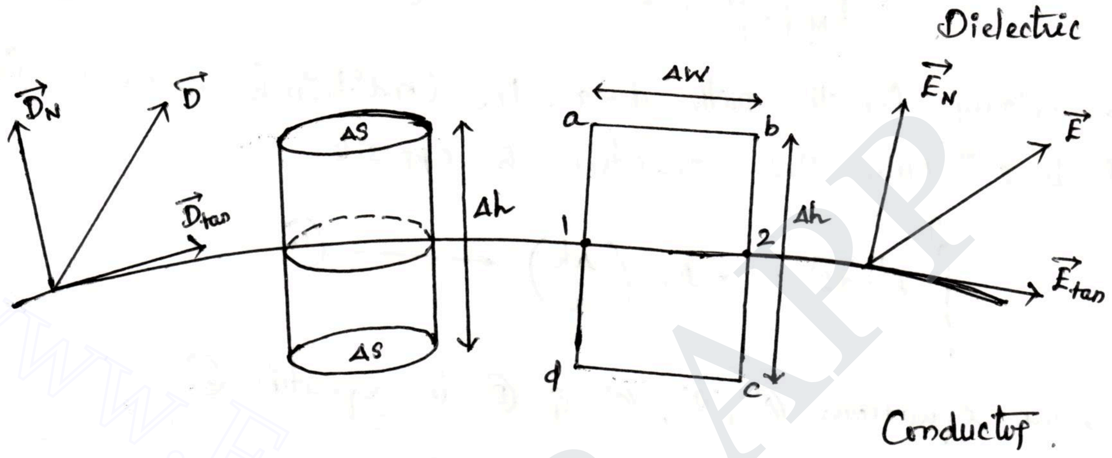
$$\boxed{E_N = \frac{\rho_S}{\varepsilon_0}} \quad \text{——— (8)}$$

(for free space $\varepsilon_r = 1$)

The flux leaves the surface normally and the normal component of flux density is equal to the surface charge density.

Boundary Conditions Between Conductor and Dielectric :

- Consider a boundary between Conductor and dielectric.
- The Conductor is ideal having infinite Conductivity. Thus $\vec{E}$, $\vec{D}$ and ρ_v are zero. ρ_s is the surface charge density on the surface of the Conductor.
- To determine the boundary Conditions the closed path (abcda) and the gaussian surface as cylinder are used.



Tangential Components at the boundary :

The workdone in carrying unit positive charge along a closed path is zero. (i,e)

$$\oint \vec{E} \cdot \vec{dl} = 0 \quad \text{——— (1)}$$

For a closed path abcda equation (1) can be written as sum of four parts.

$$\int_a^b \vec{E} \cdot \vec{dl} + \int_b^c \vec{E} \cdot \vec{dl} + \int_c^d \vec{E} \cdot \vec{dl} + \int_d^a \vec{E} \cdot \vec{dl} = 0 \quad \text{——— (2)}$$

The rectangular closed path has a height Δh and width Δw, The rectangular closed path is placed in such a way that half of it is in the Conductor and remaining half is in free space.

The portion c-d is in the Conductor medium where $\vec{E} = 0$

$$\therefore \int_c^d \vec{E} \cdot \vec{dl} = 0 \quad \text{——— (A)}$$

As Δw is very small, $\vec{E}$ over it is assumed as Constant.

(i,e) $\vec{E} = E_{tan}$ ($\because \Delta w$ is along tangential direction).

$$\int_a^b \vec{E} \cdot \vec{dl} = E_{tan} \int_a^b \vec{dl} = E_{tan}(\Delta w) \quad \text{——— (B)}$$

 As Δh is very small, $\vec{E}$ over it is assumed as Constant.

(i,e) $\vec{E} = E_N$ $\quad$ ($\because \Delta h$ is along normal direction)

$$\int_b^c \vec{E}\cdot\vec{dl} = \int_b^2 \vec{E}\cdot\vec{dl} + \int_2^c \vec{E}\cdot\vec{dl}$$

$$= E_N \int_b^2 \vec{dl} + 0 \qquad (\because \text{2-c lies in Conductor})$$

$$= E_N\left(\frac{\Delta h}{2}\right) \text{ ——— (C)}$$

Similarly for the path d-a, the Condition is same as for the path b-c, only the direction is opposite.

$$\int_d^a \vec{E}\cdot\vec{dl} = -E_N\left(\frac{\Delta h}{2}\right) \text{ ——— (D)}$$

Sub equations (A), (B), (C) & (D) in equation (2)

$$(2) \Rightarrow E_{tan}\,\Delta w + E_N\cancel{\left(\frac{\Delta h}{2}\right)} + 0 - E_N\cancel{\left(\frac{\Delta h}{2}\right)} = 0$$

$$E_{tan}\,\Delta w = 0$$

$$\boxed{E_{tan} = 0} \text{ ——— (3)}$$

WKT $\quad D_{tan} = \varepsilon E_{tan}$

Sub $E_{tan} = 0$

$$\boxed{D_{tan} = 0} \text{ ——— (4)}$$

The tangential Components of electric field Intensity and electric flux density are zero at the boundary between Conductor and dielectric.

Normal Components at the boundary:

By Gauss's law

$$\oint \vec{D}\cdot\vec{ds} = Q \text{ ——— (5)}$$

For the gaussian surface equation (5) can be written as

$$\int_{top} \vec{D}\cdot\vec{ds} + \int_{bottom} \vec{D}\cdot\vec{ds} + \int_{lateral} \vec{D}\cdot\vec{ds} = Q \text{ ——— (6)}$$

The bottom surface is in the conductor where $\vec{D'} = 0$.

$$\therefore \int_{bottom} \vec{D'} \cdot \vec{ds'} = 0 \quad \text{--- (E)}$$

For the lateral surface, the area is $2\pi r(\Delta h)$

As Δh tends to zero, area of lateral surface is zero.

$$\therefore \int_{lateral} \vec{D'} \cdot \vec{ds'} = 0 \quad \text{--- (F)}$$

For smaller top surface $\vec{D'}$ is assumed as constant.

(i,e) $\vec{D'} = D_N$

$$\therefore \int_{top} \vec{D'} \cdot \vec{ds'} = D_N \int_{top} \vec{ds'} = D_N \Delta S \quad \text{--- (G)}$$

Sub equations (E), (F) & (G) in equation (6)

$$D_N \Delta S = Q$$

$$D_N \cancel{\Delta S} = \rho_S \cancel{\Delta S}$$

$$\boxed{D_N = \rho_S} \quad \text{--- (7)}$$

WKT

$$E_N = \frac{D_N}{\varepsilon}$$

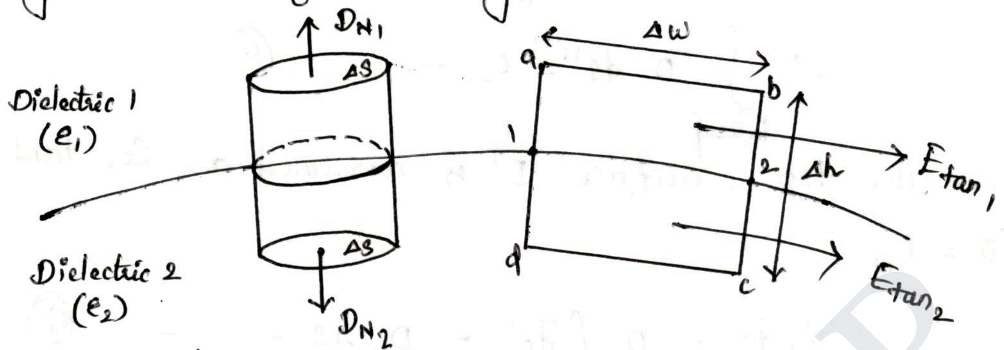
Sub $D_N = \rho_S$ & $\varepsilon = \varepsilon_0 \varepsilon_r$

$$\boxed{E_N = \frac{\rho_S}{\varepsilon_0 \varepsilon_r}} \quad \text{--- (8)}$$

The flux leaves the surface normally and the normal component of flux density is equal to the surface charge density.

BOUNDARY CONDITIONS BETWEEN TWO DIELECTRICS:

- Consider a boundary between two perfect dielectrics.
- One dielectric has permittivity ϵ_1 and the other has permittivity ϵ_2.
- To determine the boundary conditions the closed path (abcda) and the gaussian surface as cylinder are used.



Tangential Components at the boundary:

The workdone in carrying unit positive charge along a closed path is zero. (i,e)

$$\oint \vec{E} \cdot \vec{dl} = 0 \quad \text{———} \quad (1)$$

For a closed path abcda equation (1) can be written as sum of four parts.

$$\int_a^b \vec{E} \cdot \vec{dl} + \int_b^c \vec{E} \cdot \vec{dl} + \int_c^d \vec{E} \cdot \vec{dl} + \int_d^a \vec{E} \cdot \vec{dl} = 0 \quad \text{———} \quad (2)$$

The rectangle to be reduced at the surface to analyse the boundary conditions, $\Delta h \to 0$.

$$\text{As } \Delta h \to 0, \quad \int_b^c \vec{E} \cdot \vec{dl} = 0 \quad \& \quad \int_d^a \vec{E} \cdot \vec{dl} = 0 \quad \text{———} \quad (A)$$

a-b is in dielectric 1, as Δw is very small $\vec{E}$ over it is assumed as constant (i,e) $\vec{E} = E_{tan_1}$ ($\because \Delta w$ is along tangential direction)

$$\int_a^b \vec{E} \cdot \vec{dl} = E_{tan_1} \int_a^b \vec{dl} = E_{tan_1} \Delta w \quad \text{———} \quad (B)$$

c-d is in dielectric 2, hence the corresponding component of $\vec{E}$ is E_{tan_2} as c-d direction is also tangential to the surface. But the direction of c-d is opposite to a-b.

$$\therefore \int_c^d \vec{E} \cdot \vec{dl} = -E_{tan_2} \Delta w \quad \text{———} \quad (C)$$

 Sub equations Ⓐ, Ⓑ and Ⓒ in equation ②

$$E_{tan_1}\Delta w + 0 - E_{tan_2}\Delta w + 0 = 0$$

$$(E_{tan_1} - E_{tan_2})\Delta w = 0$$

$$E_{tan_1} - E_{tan_2} = 0$$

$$\boxed{E_{tan_1} = E_{tan_2}} \quad \text{---} \quad ③$$

The electric field intensity is Continuous across the boundary.

WKT $\quad D_{tan_1} = \varepsilon_1 E_{tan_1}$ & $D_{tan_2} = \varepsilon_2 E_{tan_2}$

$$③ \Rightarrow \quad \frac{D_{tan_1}}{\varepsilon_1} = \frac{D_{tan_2}}{\varepsilon_2}$$

$$\frac{D_{tan_1}}{D_{tan_2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_0 \varepsilon_{r1}}{\varepsilon_0 \varepsilon_{r2}}$$

$$\boxed{\frac{D_{tan_1}}{D_{tan_2}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}} \quad \text{---} \quad ④$$

The electric flux density is discontinuous across the boundary.

<u>Normal Components at the boundary:</u>

By Gauss's law

$$\int \vec{D}\cdot\vec{ds} = Q \quad \text{---} \quad ⑤$$

For the Gaussian surface equation ⑤ Can be written as

$$\int_{top} \vec{D}\cdot\vec{ds} + \int_{bottom} \vec{D}\cdot\vec{ds} + \int_{lateral} \vec{D}\cdot\vec{ds} = Q \quad \text{---} \quad ⑥$$

As $\Delta h \to 0$, the flux leaving from the lateral surface is Zero.

$$\int_{lateral} \vec{D}\cdot\vec{ds} = 0 \quad \text{---} \quad Ⓓ$$

For top elementary surface, flux density is assumed as Constant. (i,e) $D = D_{N_1}$

$$\int_{top} \vec{D}\cdot\vec{ds} = D_{N_1}\int \vec{ds} = D_{N_1}\Delta S \quad \text{---} \quad Ⓔ$$

 For bottom surface, $\overline{D}$ is assumed as Constant and it is opposite to the direction of D_{N_1}.

(i,e) $\overline{D} = -D_{N_2}$

$$\int_{bottom} \overline{D}\cdot\overline{ds} = -D_{N_2}\int \overline{ds} = -D_{N_2}\Delta S \longrightarrow (F)$$

Sub equations (D), (E) & (F) in eqn (6)

$$D_{N_1}\Delta S + 0 - D_{N_2}\Delta S = Q$$

$$(D_{N_1} - D_{N_2})\cancel{\Delta S} = \rho_S \cancel{\Delta S}$$

$$\boxed{D_{N_1} - D_{N_2} = \rho_S}$$

for perfect dielectric, $\rho_S = 0$

$$\therefore D_{N_1} - D_{N_2} = 0$$

$$\boxed{D_{N_1} = D_{N_2}} \longrightarrow (7)$$

Normal Components of flux density is Continuous across the boundary.

WKT $D_{N_1} = \epsilon_1 E_{N_1}$ and $D_{N_2} = \epsilon_2 E_{N_2}$

$$(7) \Rightarrow \quad \epsilon_1 E_{N_1} = \epsilon_2 E_{N_2}$$

$$\frac{E_{N_1}}{E_{N_2}} = \frac{\epsilon_2}{\epsilon_1} = \frac{\epsilon_0 \epsilon_{r_2}}{\epsilon_0 \epsilon_{r_1}}$$

$$\boxed{\frac{E_{N_1}}{E_{N_2}} = \frac{\epsilon_{r_2}}{\epsilon_{r_1}}}$$

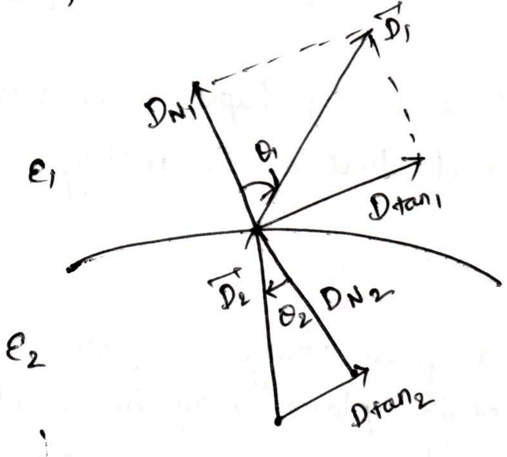
Normal Components of Electric field Intensity is discontinuous across the boundary.

Refraction of $\overline{D}$ at the boundary:

– The direction of $\overline{D}$ & $\overline{E}$ change at the boundary between the two dielectrics.

 Let $\vec{D_1}$ & $\vec{E_1}$ make an angle θ_1 with the normal to the surface. (59)

- $\vec{D_1}$ & $\vec{E_1}$ direction is same.



from the figure

$$\frac{D_{N_1}}{D_1} = \cos\theta_1$$

$$\boxed{D_{N_1} = D_1 \cos\theta_1}$$

$$\frac{D_{N_2}}{D_2} = \cos\theta_2$$

$$\boxed{D_{N_2} = D_2 \cos\theta_2}$$

WKT $\quad D_{N_1} = D_{N_2}$

$$\boxed{D_1 \cos\theta_1 = D_2 \cos\theta_2}$$

from the figure

$$\sin\theta_1 = \frac{D_{tan1}}{D_1} \Rightarrow \boxed{D_{tan_1} = D_1 \sin\theta_1}$$

$$\sin\theta_2 = \frac{D_{tan2}}{D_2} \Rightarrow \boxed{D_{tan_2} = D_2 \sin\theta_2}$$

$$\tan\theta_1 = \frac{D_{tan1}}{D_{N_1}} \quad \text{and} \quad \tan\theta_2 = \frac{D_{tan2}}{D_{N2}}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{\cancel{D_{N_1}}} \times \frac{\cancel{D_{N2}}}{D_{tan2}} \qquad \because D_{N_1} = D_{N2}$$

$$= \frac{\epsilon_1}{\epsilon_2} \qquad \because \frac{D_{tan1}}{D_{tan_2}} = \frac{\epsilon_1}{\epsilon_2}$$

$$\boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}} \Rightarrow \text{Law of refraction.}$$

If $\epsilon_1 > \epsilon_2$, then $\theta_1 > \theta_2$.

(i) $\vec{D}$ is larger in the region of larger permittivity.

(ii) $\vec{E}$ is larger in the region of smaller permittivity.

(iii) $|\vec{D_1}| = |\vec{D_2}|$ if $\theta_1 = \theta_2 = 0°$

(iv) $|\vec{E_1}| = |\vec{E_2}|$ if $\theta_1 = \theta_2 = 90°$

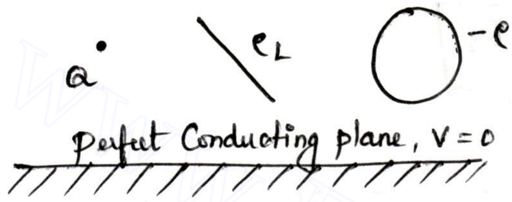
METHOD OF IMAGES:

— Used to determine V, $\vec{E}$, $\vec{D}$ and ρ_s due to charges in the presence of conductors.

— By this method solving poisson's and Laplace's equation can be avoided but rather utilize the fact that a conducting surface is an equipotential.

Image Theory:

Image Theory states that a given charge configuration above an infinite grounded perfect conducting plane may be replaced by the charge configuration itself, its image and an equipotential surface in place of the conducting plane.



Charge Configuration above a perfectly conducting plane

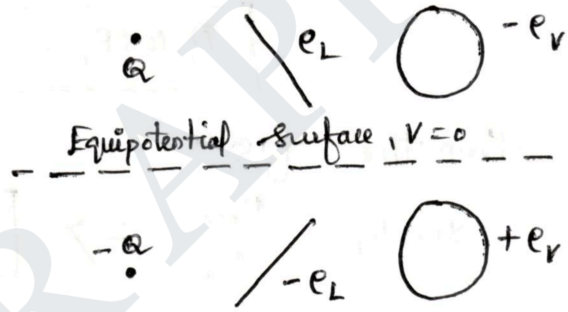


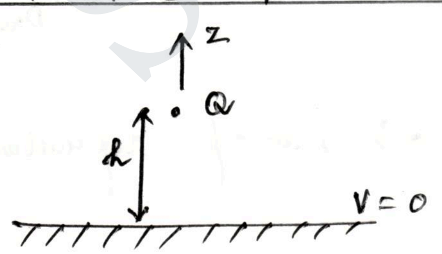
Image Configuration with the conducting plane replaced by equipotential surface.

To apply image method, two conditions must be satisfied:

(i) The image charges must be located in the conducting region

(ii) The image charges must be located such that on the conducting surface the potential is zero or constant.

(a) A POINT CHARGE ABOVE A GROUNDED CONDUCTING PLANE:



Point charge in conducting plane

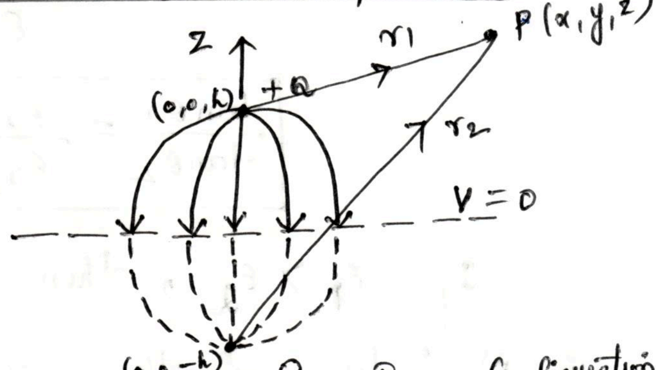


Image Configuration

— Consider a point charge Q placed at a distance h from a perfect conducting plane of infinite extent.

— The Image Configuration is shown in figure.

– The Electric field Intensity in the region above the plane at point $P(x, y, z)$ is given by

$$\vec{E} = \vec{E_+} + \vec{E_-} \quad \text{——— (1)}$$

$$\vec{E} = \left[\frac{Q}{4\pi\varepsilon_0 r_1^2}\vec{u_{r_1}}\right] + \left[\frac{-Q}{4\pi\varepsilon_0 r_2^2}\vec{u_{r_2}}\right]$$

$$\vec{E} = \frac{Q\vec{r_1}}{4\pi\varepsilon_0 r_1^3} - \frac{Q\vec{r_2}}{4\pi\varepsilon_0 r_2^3} \quad \text{——— (2)}$$

$$\vec{r_1} = (x-0)\vec{a_x} + (y-0)\vec{a_y} + (z-h)\vec{a_z} = x\vec{a_x} + y\vec{a_y} + (z-h)\vec{a_z}$$

$$\vec{r_2} = (x-0)\vec{a_x} + (y-0)\vec{a_y} + (z+h)\vec{a_z} = x\vec{a_x} + y\vec{a_y} + (z+h)\vec{a_z}$$

$$(2) \Rightarrow \vec{E} = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{x\vec{a_x} + y\vec{a_y} + (z-h)\vec{a_z}}{[x^2+y^2+(z-h)^2]^{3/2}} - \frac{x\vec{a_x} + y\vec{a_y} + (z+h)\vec{a_z}}{[x^2+y^2+(z+h)^2]^{3/2}}\right\} \text{——— (3)}$$

when $z=0$, $\vec{E}$ has only z-component (i.e) $\vec{E}$ is normal to the conducting surface.

– The potential in the region above the plane at point $P(x, y, z)$ is given by,

$$V = V_+ + V_- \quad \text{——— (4)}$$

$$= \frac{Q}{4\pi\varepsilon_0 r_1} - \frac{Q}{4\pi\varepsilon_0 r_2}$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left\{\frac{1}{[x^2+y^2+(z-h)^2]^{1/2}} - \frac{1}{[x^2+y^2+(z+h)^2]^{1/2}}\right\} \text{——— (4)}$$

When $z=0$, $V=0$.

The surface charge density of the induced charge is

$$\rho_s = D_n = \varepsilon_0 [E_n]\Big|_{z=0}$$

$$= \varepsilon_0 \times \frac{Q}{4\pi\varepsilon_0} \times \frac{-2h}{(x^2+y^2+h^2)^{3/2}}$$

$$\rho_s = \frac{-Qh}{2\pi(x^2+y^2+h^2)^{3/2}} \quad \text{——— (5)}$$

The total induced charge on the Conducting plane is;

$$Q_i = \int_S \rho_s \, ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-Qh}{2\pi.(x^2+y^2+h^2)^{3/2}} \, dx\,dy$$

By changing the variable from Cartesian to Cylindrical,

$$x^2+y^2=\rho^2 \quad , \quad dx\,dy = \rho\, d\rho\, d\phi$$

$$\Rightarrow Q_i = \frac{-Qh}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\rho\, d\rho\, d\phi}{(\rho^2+h^2)^{3/2}}$$

$$= \frac{-Qh}{2\pi}\int_0^{\infty} [\phi]_0^{2\pi} (\rho^2+h^2)^{-3/2} \times \frac{1}{2} d(\rho^2)$$

$$\because \rho\, d\rho = \frac{1}{2} d(\rho^2)$$

$$= \frac{-Qh}{\cancel{2\pi}} \times \cancel{2\pi} \times \frac{1}{\cancel{2}}\left[\frac{-\cancel{2}}{(\rho^2+h^2)^{1/2}}\right]_0^{\infty}$$

$$= -Q\cancel{h}\left[\frac{1}{\cancel{h}}\right]$$

$$\boxed{Q_i = -Q}$$

— All the flux lines terminating on the Conductor would have terminated on the image charge if the Conductor were absent.

(b) <u>A LINE CHARGE ABOVE A GROUNDED CONDUCTING PLANE</u>:

— Consider an infinite line charge ρ_L C/m located at a distance h from the grounded conducting plane $z=0$.

— The infinite line charge ρ_L may be assumed to be at $x=0$, $z=h$ and the image $-\rho_L$ at $x=0$, $z=-h$. So that the two are parallel to the y-axis

— The electric field at point P is given by,

$$\vec{E} = \vec{E}_+ + \vec{E}_- \quad \text{——— (1)}$$

$$= \left[\frac{\rho_L}{2\pi\varepsilon_0 \rho_1}\vec{u}_{\rho_1}\right] + \left[\frac{-\rho_L}{2\pi\varepsilon_0\rho_2}\vec{u}_{\rho_2}\right]$$

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0}\left[\frac{\vec{\rho_1}}{\rho_1^2} - \frac{\vec{\rho_2}}{\rho_2^2}\right] \longrightarrow (2)$$

$$\vec{\rho_1} = (x-0)\vec{a_x} + (y-y)\vec{a_y} + (z-h)\vec{a_z} = x\vec{a_x} + (z-h)\vec{a_z}$$

$$\vec{\rho_2} = (x-0)\vec{a_x} + (y-y)\vec{a_y} + (z+h)\vec{a_z} = x\vec{a_x} + (z+h)\vec{a_z}$$

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0}\left[\frac{x\vec{a_x} + (z-h)\vec{a_z}}{x^2+(z-h)^2} - \frac{x\vec{a_x} + (z+h)\vec{a_z}}{x^2+(z+h)^2}\right] \longrightarrow (3)$$

when $z=0$, $\vec{E}$ has only z-component. So $\vec{E}$ is normal to the Conducting Surface.

The potential at point P is

$$V = V_+ + V_-$$

$$= \left[\frac{\rho_L}{2\pi\varepsilon_0}\ln\rho_1\right] + \left[\frac{-\rho_L}{2\pi\varepsilon_0}\ln\rho_2\right]$$

$$V = \frac{\rho_L}{2\pi\varepsilon_0}\ln\left(\frac{\rho_1}{\rho_2}\right) \longrightarrow (4)$$

Sub $\rho_1 = \sqrt{x^2+(z-h)^2}$ & $\rho_2 = \sqrt{x^2+(z+h)^2}$.

$$V = \frac{\rho_L}{2\pi\varepsilon_0}\ln\left[\frac{x^2+(z-h)^2}{x^2+(z+h)^2}\right]^{1/2}$$

If $z=0$, $V=0$.

The surface charge induced on the Conducting plane is

$$\rho_s = D_n = \varepsilon_0 E_n\Big|_{z=0} = \varepsilon_0 \times \frac{\rho_L}{2\pi\varepsilon_0} \times \frac{-2h}{x^2+h^2}$$

$$\rho_s = \frac{-\rho_L h}{\pi(x^2+h^2)} \longrightarrow (5)$$

The induced Charge per length on the Conducting plane is

$$\rho_i = \int_s \rho_s\, ds = \frac{-\rho_L h}{\pi}\int_{-\infty}^{\infty}\frac{dx}{x^2+h^2}$$

let $x = h \tan \alpha$. $x = -\infty \Rightarrow \alpha = -\pi/2$

$dx = h \sec^2 \alpha \, d\alpha$. $x = \infty \Rightarrow \alpha = \pi/2$.

$$\rho_i = -\frac{\rho_L h}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\cancel{h} \cancel{\sec^2} \alpha \, d\alpha}{h^{\cancel{2}} \cancel{\sec^2} \alpha}$$

$$= \frac{-\rho_L \cancel{h}}{\pi} \times \frac{1}{\cancel{h}} \left[\alpha\right]_{-\pi/2}^{\pi/2}$$

$$= -\frac{\rho_L}{\cancel{\pi}} \times \cancel{\pi}$$

$$\boxed{\rho_i = -\rho_L}$$

CURRENT (I):

– The rate of flow of charge at a specified point or across a specified surface is called electric current.

– It is measured in the unit Ampere (A)

$$\boxed{I = \frac{dQ}{dt}} \quad \text{Ampere.}$$

Drift Current:

The current exist in the conductors, due to the drifting of electrons under the influence of the applied voltage is called drift current.

Displacement Current:

The flow of charges in the dielectrics, under the influence of the electric field Intensity is called as displacement current.

CURRENT DENSITY ($\vec{J}$):

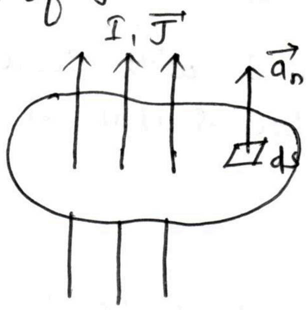
– The current passing through the unit surface area, when the surface is held normal to the direction of current is called as current density.

– It is measured in Ampere per square meters. (A/m^2)

RELATION BETWEEN CURRENT AND CURRENT DENSITY:

– Consider the surface S and I is the current passing through the surface.

– The direction of current is normal to the surface S and hence direction of $\vec{J}$ is also normal to the surface S.



– Consider a incremental surface area ds and $\vec{a_n}$ is the unit normal vector to the incremental surface ds.

$$\vec{ds} = ds\,\vec{a_n}$$

$$\vec{J} = J\,\vec{a_n}$$

The differential current dI passing through the differential surface ds is the dot product of $\vec{J}$ and $\vec{ds}$

$$dI = \vec{J} \cdot \vec{ds}$$

$$= J\vec{a_n} \cdot ds\,\vec{a_n}$$

$$dI = J\,dS$$

$$\boxed{I = \int J\,dS}$$

If $\vec{J}$ is not normal to $\vec{ds}$,

$$\boxed{I = \int \vec{J} \cdot \vec{ds}}$$

Relation between $\vec{J}$ and ρ_v:

The relation between $\vec{J}$ and ρ_v is given by

$$\boxed{\vec{J} = \rho_v \vec{v}}$$

where $\vec{J}$ – Current density
ρ_v – Volume charge density
$\vec{v}$ – Velocity vector.

Continuity Equation:

– Continuity equation is based upon the principle of conservation of charge. (i,e) "charges can neither be created nor be destroyed".

– Consider a closed surface S with current density $\vec{J}$ and the total current I crossing the surface S is

$$I = \oint \vec{J} \cdot \vec{ds}$$

– Current flows outwards from the closed surface.

– The outward rate of flow of charge gets balanced by the rate of decrease of charge inside the closed surface.

$Q_i \rightarrow$ Charge within the closed surface.

$-\frac{dQ_i}{dt} \rightarrow$ Rate of decrease of charge inside the closed surface.

Due to principle of Conservation of charge, the rate of decrease is same as rate of outward flow of charge.

$$I = -\frac{dQ_i}{dt}$$

$$\boxed{\int \vec{J} \cdot \vec{dS} = -\frac{dQ_i}{dt}} \Rightarrow \text{Integral form of Continuity equation.}$$

By divergence theorem

$$\int_V (\nabla \cdot \vec{J})\, dV = -\frac{dQ_i}{dt}$$

$$\text{WKT } Q_i = \int_V \rho_v \, dV$$

$$\Rightarrow \int_V (\nabla \cdot \vec{J})\, dV = -\frac{d}{dt}\int \rho_v \, dV$$

$$\int_V \nabla \cdot \vec{J}\, dV = -\int \frac{\partial \rho_v}{\partial t}\, dV$$

$$\boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}} \Rightarrow \text{point form or differential form of Continuity equation.}$$

The equation states that the current or the charge per second, diverging from a small volume per unit volume is equal to the rate of decrease of charge per unit volume at every point.

Steady Current

The steady currents are not the function of time.

(i.e) $\frac{\partial \rho_v}{\partial t} \rightarrow 0$

$\therefore$ Continuity equation becomes

$$\boxed{\nabla \cdot \vec{J} = 0}$$

The steady currents have no sources or sinks, as it is constant.

CONDUCTORS:

- In conductors, under the effect of applied electric field, the available free electrons start moving. The moving electrons strike the adjacent atoms and rebound in the random directions. This is called drifting of electrons.
- After some time, the electrons attain the constant average velocity called drift velocity (v_d).
- The current constituted due to the drifting of such electrons in metallic conductors is called drift current.
- The drift velocity is directly proportional to the applied electric field ($\vec{E}$).

$$\vec{v_d} \propto \vec{E}$$

$$\vec{v_d} = -\mu_e \vec{E} \quad \text{——— (1)}$$

where $\mu_e \rightarrow$ mobility of the electrons

-ve sign indicates that the velocity of electrons is against the direction of field $\vec{E}$.

From the relation between $\vec{J}$ and $\vec{v}$,

$$\vec{J} = \rho_e \vec{v_d} \quad \text{——— (2)}$$

$\rho_e \rightarrow$ charge density due to free electrons.

Sub (1) in (2)

$$\boxed{\vec{J} = -\rho_e \mu_e \vec{E}} \quad \text{——— (3)}$$

Point form of Ohm's law

For a metallic Conductor, the relationship between $\vec{J}$ and $\vec{E}$ can be expressed interms of Conductivity of the material as

$$\boxed{\vec{J} = \sigma \vec{E}} \quad \text{——— (4)}$$

where $\sigma \rightarrow$ Conductivity of the material

By Comparing (3) & (4)

$$\boxed{\sigma = -\rho_e \mu_e} \quad \text{——— (5)}$$

Properties of Conductor:

(i) Under Static Conditions, no charge and no electric field can exist at any point within the Conducting material.

(ii) The charge can exist on the surface of the Conductor giving rise to surface charge density.

(iii) Within a Conductor, the charge density is always zero.

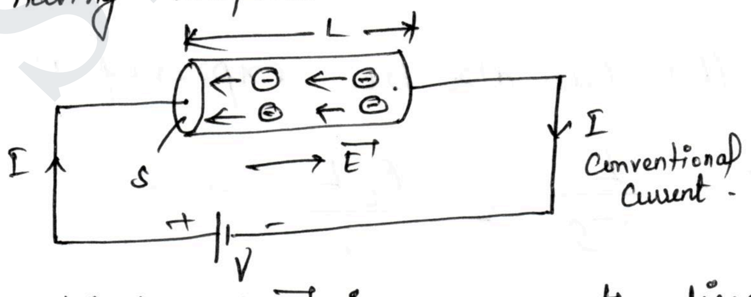
(iv) The charge distribution on the surface depends on the shape of the surface.

(v) The Conductivity of an ideal Conductor is infinite.

(vi) The Conductor surface is an equipotential surface.

RESISTANCE OF A CONDUCTOR:

– Consider that the voltage V is applied to a Conductor of length L having uniform cross section.



– The direction of $\vec{E}$ is same as the direction of Conventional Current, which is opposite to the flow of electrons.

– The electric field applied is uniform and its magnitude is given by,

$$E = \frac{V}{L} \quad \text{——— (1)}$$

 The Conductor has uniform cross section S

$$I = \int_S \vec{J} \cdot \vec{ds} = JS \quad \text{——} \quad (2)$$

The current direction is normal to the surface S

$$J = \frac{I}{S} = \sigma E \quad \text{——} \quad (3)$$

Sub (1) in (3)

$$J = \frac{\sigma V}{L}$$

$$V = \frac{JL}{\sigma}$$

Sub $J = \frac{I}{S}$

$$V = \frac{IL}{\sigma S}$$

$$V = \left(\frac{L}{\sigma S}\right) I$$

$$\boxed{R = \frac{V}{I} = \frac{L}{\sigma S}} \quad \text{——} \quad (4)$$

$\Rightarrow$ for uniform field.

for Nonuniform field,

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \vec{E} \cdot \vec{dl}}{\int_S \vec{J} \cdot \vec{ds}} = \frac{-\int_b^a \vec{E} \cdot \vec{dl}}{\int_S \sigma \vec{E} \cdot \vec{ds}}$$

$\Rightarrow$ for nonuniform field.

The resistance (R) can also be expressed as

$$R = \frac{L}{\sigma S} = \frac{\rho_c L}{S} \ \Omega$$

where $\rho_c = \frac{1}{\sigma}$ = Resistivity of the Conductor in Ωm.

DIELECTRIC MATERIALS:

– The dielectric materials do not have free charges. The charges in dielectrics are bound by the finite forces and hence called bound charges.

– The bound charges cannot contribute to the conduction process. When electric field $\vec{E}$ is applied, they shift their relative positions. This shift in the relative positions of bound charges, allows the dielectric to store energy.

– The shifts in positive and negative charges are in opposite directions and under the influence of an applied electric field such charges act like small electric dipoles.

Dielectrics

Non polar

– dipole arrangement is totally absent, in absence of electric field $\vec{E}$.

Ex: Hydrogen, Oxygen

Polar

– dipole arrangement exist without application of $\vec{E}$.

Ex: Water, Sulphur dioxide.

POLARIZATION:

"Separation of bound charges to produce electric dipoles, under the influence of electric field $\vec{E}$ is called as Polarization."

When the dipole is formed due to polarization, there exist an electric dipole moment $\vec{p}$.

$$\therefore \vec{p} = Q\vec{d}$$

Q – Magnitude of one of two charges

$\vec{d}$ – distance vector from negative to positive charge.

By Superposition principle.

$$\vec{P}_{total} = Q_1\vec{d_1} + Q_2\vec{d_2} + \dots + Q_n\vec{d_n}$$

$$\vec{P}_{total} = \sum_{i=1}^{n\Delta V} Q_i\vec{d_i}$$

Defn/

The Polarization is defined as the total dipole moment per unit volume.

$$\vec{P} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n\Delta V} Q_i \vec{d_i}}{\Delta V} \quad C/m^2 \quad \text{——— (A)}$$

The polarization increases the electric flux density in dielectric medium.

$$\therefore \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \quad \text{——— (B)}$$

for isotropic and linear medium $\vec{P}$ & $\vec{E}$ are parallel and they are related as

$$\vec{P} = \chi_e \varepsilon_0 \vec{E} \quad \text{——— (C)}$$

χ_e – electric susceptibility.

Sub (C) in (B)

$$\vec{D} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E}$$

$$\vec{D} = \varepsilon_0 (1+\chi_e) \vec{E} \quad \text{——— (D)}$$

WKT $\vec{D} = \varepsilon_0 \varepsilon_r \vec{E}$ ——— (E)

Comparing (D) & (E)

$$\boxed{\varepsilon_r = 1 + \chi_e} \quad \text{——— (F)}$$

$\Rightarrow$ Relative Permittivity (or) Dielectric Constant

Properties of Dielectric materials:

(i) The dielectrics do not contain any free charges.

(ii) Due to polarization, the dielectrics can store energy.

(iii) The electric field outside and inside the dielectrics get modified due to the induced electric dipoles.

(iv) The induced dipoles produce their own electric field and align in the direction of the applied electric field.

Unit-3 EMF

Magnetostatics.

- Lorentz Force Equation
- Law of no magnetic Monopoles.
- Ampere's Law
- Vector Magnetic potential
- Biot-Savart law and applications
- Magnetic Field Intensity and Idea of Relative permeability
- Magnetic Circuits
- Behaviour of Magnetic Materials
- Boundary Conditions
- Inductance and Inductors
- Magnetic Energy
- Magnetic Forces and Torques.

Introduction:

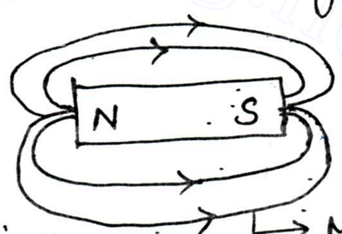
i) Magnetostatics:

The direct current (d.c) is a stedy flow of current and the magnetic field produced by a conductor carrying d.c current is a static steady magnetic field.

The study of steady magnetic field which is produced due to the flow of direct current through a conductor is called magnetostatics.

ii) Magnetic flux lines:

An Imaginary lines around the magnet are called magnetic lines of force (or) magnetic flux lines. The direction of such lines is always from N pole to S pole, external to the magnet.



→ Magnetic lines of Force.

Electric Flux lines	Magnetic Flux Lines.
i) Electric flux lines are produced even if only one charges (either +ve or −ve) exist.	Magnetic flux lines are produced only when a pair of poles (N or S pole) exist.

The magnetic flux lines always form a closed loop.

iii) Magnetic flux:

The total number of magnetic lines of force is called a magnetic flux denoted as ϕ. It is measured in weber (wb)

iv) Magnetic Flux Density ($\vec{B}$):

The total magnetic lines of force (i.e) magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density.

- It is denoted by '$\vec{B}$'
- It is a vector quantity
- Unit of $\vec{B}$ is wb/m^2 (or) Tesla

v) Magnetic Field Intensity ($\vec{H}$):

The magnetic field Intensity at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength.

- It is denoted as ($\vec{H}$)
- Vector Quantity.
- Unit of $\vec{H}$ is N/wb (or) A/m

vi) Relation between $\vec{B}$ and $\vec{H}$.

The relation between $\vec{B}$ and $\vec{H}$ is

$$\vec{B} = \mu \vec{H}$$

where

μ - permeability

$\mu = \mu_0 \mu_r$

$\mu_0 = 4\pi \times 10^{-7}$ (H/m)

μ_r = Relative permeability.

Lorentz Force Equation:

When a charge 'Q' is placed in an electric field 'E', it experiences an electric force ($\vec{F_e}$)

$$\vec{F_e} = Q\vec{E} \longrightarrow (1)$$

$\vec{E}$ - Electric Field Intensity.

When a moving charge 'Q' is placed in a steady magnetic field experiences magnetic force ($\vec{F_m}$)

$$\vec{F_m} = Q(\vec{v} \times \vec{B}) \longrightarrow (2)$$

$\vec{F_m}$ is directly proportional to the magnitudes of Q, $\vec{v}$ and $\vec{B}$ and also the sine of the angle between $\vec{v}$ and $\vec{B}$. ($\vec{v} \times \vec{B}$ ↳ cross product)

The direction of $\vec{F_m}$ is perpendicular to the plane containing $\vec{v}$ and $\vec{B}$ both.

From eqn (1) $\vec{Fe}$ is independent of the velocity of charges moving. Therefore electric force performs work on the charge.

From eqn (2) $\vec{Fm}$ is dependent on the velocity of moving charge.

The total force on a moving charge in the presence of both electric and magnetic field is

$$\vec{F} = \vec{Fe} + \vec{Fm}$$

$$= Q\vec{E} + Q(\vec{v} \times \vec{B})$$

$$\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})] \rightarrow (3)$$

This equation is called Lorentz force equation. The solution of this equation is useful in the determination of

(i) electron orbits in magnetron.

(ii) Proton paths in cylotron

iii) plasma characteristics in magnetohydrodynamic generator (HHD)

If the mass of the charge is m

$$\vec{F} = m\vec{a}$$

$$Q[\vec{E} + (\vec{v} \times \vec{B})] = m\frac{d\vec{v}}{dt} \rightarrow (4)$$

Problem:

1. A point charge of $Q = -1.2C$ has velocity $\vec{v} = (5\vec{a_x} + 2\vec{a_y} - 3\vec{a_z})$ m/s. Find the magnitude of the force exerted on the charge if

a) $\vec{E} = -18\vec{a_x} + 5\vec{a_y} - 10\vec{a_z}$ V/m

b) $\vec{B} = -4\vec{a_x} + 4\vec{a_y} + 3\vec{a_z}$ T

c) Both are present simultaneously.

Soln:

(a) $\vec{F_e} = Q\vec{E}$

$$= -1.2[-18\vec{a_x} + 5\vec{a_y} - 10\vec{a_z}]$$

$$\vec{F_e} = 21.6\vec{a_x} - 6\vec{a_y} + 12\vec{a_z}.$$

Magnitude of $\vec{F_e} = |\vec{F_e}|$

$$F_e = \sqrt{(21.6)^2 + (-6)^2 + (12)^2}$$

$$\boxed{F_e = |\vec{F_e}| = 25.4275\,N}$$

b) $\vec{F_m} = Q[\vec{v} \times \vec{B}]$

$$\vec{v} \times \vec{B} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ 5 & 2 & -3 \\ -4 & 4 & 3 \end{vmatrix}$$

$$= 18\vec{a_x} - 3\vec{a_y} + 28\vec{a_z}$$

$$\vec{F_m} = -1.2[18\vec{a_x} - 3\vec{a_y} + 28\vec{a_z}]$$

$$\vec{F_m} = -21.6\vec{a_x} + 3.6\vec{a_y} - 33.6\vec{a_z}$$

Magnitude of $\vec{Fm}$

$$F_m = |\vec{F_m}| = \sqrt{(-21.6)^2+(3.6)^2+(-33.6)^2}$$

$$\boxed{F_m = 40.1058\ N}$$

(c) $\vec{F} = \vec{Fe} + \vec{Fm}$

$$= 21.6\,\vec{a_x} - 6\,\vec{a_y} + 12\,\vec{a_z} - 21.6\,\vec{a_x} + 3.6\,\vec{a_y} - 33.6\,\vec{a_z}$$

$$\boxed{\vec{F} = -2.4\,\vec{a_y} - 21.6\,\vec{a_z}}$$

magnitude of $\vec{F}$

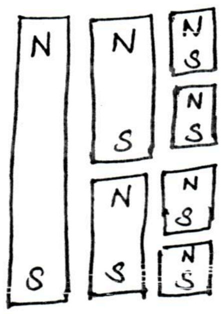
$$F = |\vec{F}| = \sqrt{(-2.4)^2+(-21.6)^2}$$

$$\boxed{F = 21.7329\ N}$$

Law of no magnetic Monopoles:

No magnetic monopole means an isolated magnetic pole cannot exist.

Consider a bar magnet with north and south pole as shown in figure. If this magnet is cut into two segments, new south and north poles appear. If each of the two shorter magnet is cut again into two segments, we have four magnets, each with a ~~north~~ pole.



Successive division of Bar magnet.

and south pole. This process could be continued until the magnets are of atomic dimensions. Obviously, magnetic poles cannot be isolated.

The magnetic flux lines follow closed paths from one end of a magnet to the other end outside the magnet.

We know that, the flux in weber passing through an unit area.

(i.e) $\vec{B} = \dfrac{\phi}{S}$

$$\phi = \vec{B} S$$

In differential form

$$d\phi = \vec{B} \cdot d\vec{s}$$

$$\boxed{\phi = \oint \vec{B} \cdot ds}$$

In a closed surface, no. of magnetic lines of force entering must be equal to no. of magnetic flux lines leaving.

(i.e) It is referred as law of conservation of magnetic flux (or) Gauss law for magnetism.

Gauss law for magnetism states that the total outward magnetic flux through any closed surface is zero.

$$\boxed{\oint_S \vec{B}\cdot ds = 0} \rightarrow (1)$$

In Differential form,

By applying divergence theorem, equation (1) will be

$$\boxed{\nabla\cdot\vec{B} = 0} \rightarrow (2).$$

The divergence of magnetic flux density is always zero. This is called Gauss law in differential form for magnetic fields.

Ampere's circuital law (or) Ampere's work law (or) Ampere's law.

The line Integral of $\vec{H}$ around a closed path is same as the net current (I_{enc}) enclosed by the path.

$$\oint \vec{H}\cdot dl = I_{enc}.$$

(i.e) Circulation of $\vec{H}$ equals to I_{enc}.

It is applicable to determine $\vec{H}$ when the current distribution is symmetrical.

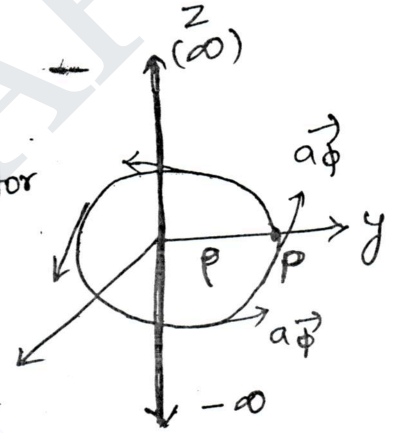
Proof:

Consider a long straight conductor carrying direct current I placed along z-axis.

- Consider a closed circular path of radius ρ which encloses the straight conductor carrying direct current I.

- The point P is at a perpendicular distance ρ from the conductor. Consider $\vec{dl}$ at P which is in $\vec{a_\phi}$ direction, tangential to circular path at P.

$$\vec{dl} = \rho d\phi \vec{a_\phi}$$



$\vec{H}$ for a Infinitely long conductor is

$$\vec{H} = \frac{I}{2\pi\rho} \vec{a_\phi} \quad A/m \longrightarrow (1)$$

$$\vec{H} \cdot \vec{dl} = \frac{I}{2\pi\cancel{\rho}} \vec{a_\phi} \cdot \cancel{\rho} d\phi \vec{a_\phi}$$

$$= \frac{I}{2\pi} d\phi$$

$$\int \vec{H} \cdot \vec{dl} = \int_0^{2\pi} \frac{I}{2\pi} d\phi$$

$$= \frac{I}{2\pi} [\phi]_0^{2\pi}$$

$$= \frac{I}{\cancel{2\pi}} \times \cancel{2\pi}$$

$$\boxed{\int \vec{H} \cdot \vec{dl} = I}$$

Point form of Ampere's circuital law:

Ampere's circuital law is

$$\oint \vec{H} \cdot \vec{dl} = I$$

By Stoke's Theorem

$$\int_S (\nabla \times \vec{H}) \cdot \vec{ds} = I \longrightarrow (A)$$

WKT

$$\int_S J \cdot ds = I \longrightarrow (B)$$

From (A) & (B)

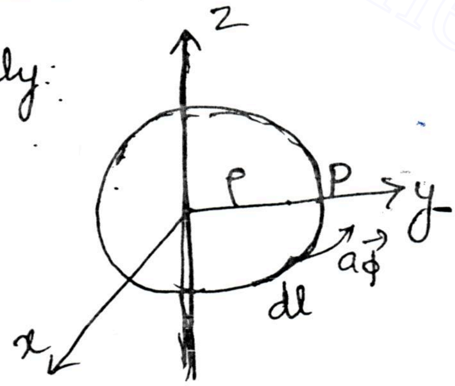
$$\boxed{\nabla \times \vec{H} = \vec{J}}$$

Here $\nabla \times \vec{H} \neq 0$, so the magnetic field is not a conservative field.

Applications of Ampere's circuital law.

1. $\vec{H}$ due to Infinitely long straight conductor.

Here $\vec{H}$ has component only in $\vec{a_\phi}$



$$\vec{H} = H_\phi \vec{a_\phi}$$

$$\vec{dl} = \rho \, d\phi \, \vec{a_\phi}$$

$$\vec{H} \cdot \vec{dl} = H_\phi \rho \, d\phi$$

By Ampere's circuital law

$$\oint \vec{H} \cdot \vec{dl} = I$$

$$\int_0^{2\pi} H_\phi \rho \, d\phi = I$$

$$H_\phi \rho \left[\phi\right]_0^{2\pi} = I$$

$$H_\phi \rho \, 2\pi = I$$

$$H_\phi = \frac{I}{2\pi\rho} \quad ; \quad \text{WKT } \vec{H} = H_\phi \vec{a_\phi}$$

$$\boxed{\therefore \vec{H} = \frac{I}{2\pi\rho} \vec{a_\phi}} \text{ A/m.}$$

$$\boxed{\vec{B} = \frac{\mu I}{2\pi\rho} \vec{a_\phi}} \text{ wb/m}^2$$

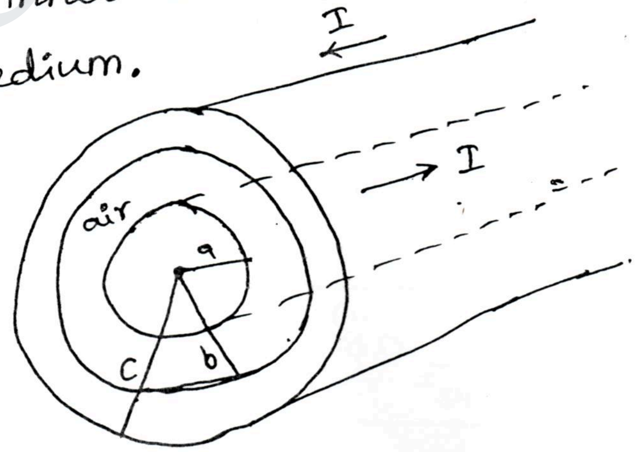
2. $\vec{H}$ due to Coaxial cable:

Consider a coaxial cable which has the radius of inner conductor 'a', inner radius of outer conductor 'b' and outer radius of outer conductor 'c'.

The current flowing through the cable is I.

The current flowing in inner and outer conductor are in opposite direction.

The inner and outer conductors are separated by air medium.

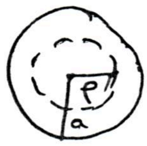


(Consider the coaxial cable lying on z-axis.

* If the conductor is in z-axis then $\vec{H}$ is along $\vec{a_\phi}$ direction)

Region (i) $\rho < a$.

- The area of cross section enclosed is $\pi\rho^2$



- The total current flowing is I through the area πa^2.
- The current enclosed by the closed path is

$$I'_{enc} = \frac{\pi\rho^2}{\pi a^2} I = \frac{\rho^2}{a^2} I.$$

Here

$$\vec{H} = H_\phi \vec{a_\phi} \quad ; \quad \vec{dl} = \rho \, d\phi \, \vec{a_\phi}$$

$$\vec{H} \cdot \vec{dl} = \rho H_\phi \, d\phi$$

By Ampere's circuital law

$$\oint \vec{H} \cdot \vec{dl} = I_{enc}.$$

$$\int_0^{2\pi} H_\phi \rho \, d\phi = \frac{\rho^2}{a^2} I$$

$$H_\phi \rho [\phi]_0^{2\pi} = \frac{\rho^2}{a^2} I$$

$$2\pi \cancel{\rho} H_\phi = \frac{\rho^{\cancel{2}}}{a^2} I$$

$$H_\phi = \frac{\rho}{2\pi a^2} I$$

$$\boxed{\vec{H} = \frac{I\rho}{2\pi a^2} \vec{a_\phi}} \text{ A/m}$$

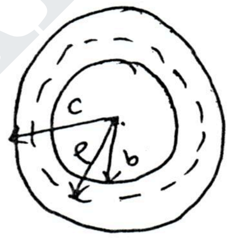
(ii) Region (ii) $a < \rho < b$.

Consider a circular path which encloses the inner conductor carrying direct current I.

The medium between inner and outer conductor is air. Therefore this case is similar to infinitely long conductor lying on z-axis.

$$\boxed{\vec{H} = \frac{I}{2\pi\rho}\vec{a_\phi}} \text{ A/m}$$

Region (iii) $b < \rho < c$



The current enclosed by the closed path is only the part of the current $(-I)$ in the outer conductor.

The total current $(-I)$ is flowing through the cross section $\pi(c^2-b^2)$ while the closed path encloses the cross section $\pi(\rho^2-b^2)$

The Total current enclosed by the closed path of outer conductor is

$$I' = \frac{\cancel{\pi}(\rho^2-b^2)}{\cancel{\pi}(c^2-b^2)}(-I)$$

$$I' = -\frac{I(\rho^2-b^2)}{(c^2-b^2)}$$

Let $I'' = I$ = current in the inner conductor

Total current enclosed by the closed path is

$$I_{enc} = I' + I''.$$

$$I_{enc} = I' + I''$$

$$= \frac{-I(\rho^2 - b^2)}{(c^2 - b^2)} + I.$$

$$= I\left[\frac{-\rho^2 + \cancel{b^2} + c^2 - \cancel{b^2}}{c^2 - b^2}\right]$$

$$I_{enc} = I\,\frac{(c^2 - \rho^2)}{(c^2 - b^2)}$$

By Ampere's circuital law

$$\oint \vec{H}\cdot d\vec{l} = I_{enc}.$$

$$\int_0^{2\pi} H_\phi\, \rho\, d\phi = I\,\frac{(c^2-\rho^2)}{(c^2-b^2)}$$

$$H_\phi\, \rho\, (\phi)_0^{2\pi} = I\,\frac{(c^2-\rho^2)}{(c^2-b^2)}$$

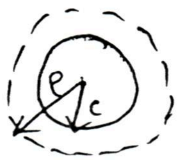
$$H_\phi\, \rho\,(2\pi) = I\,\frac{(c^2-\rho^2)}{(c^2-b^2)}$$

$$H_\phi = \frac{I}{2\pi\rho}\,\frac{(c^2-\rho^2)}{c^2-a^2}. \qquad \left[\because \text{WKT } \vec{H} = H_\phi \vec{a_\phi}\right]$$

$$\boxed{\therefore\ \vec{H} = \frac{I}{2\pi\rho}\,\frac{(c^2-\rho^2)}{(c^2-a^2)}\,\vec{a_\phi}}\ \text{A/m}$$

Region (iv) $\rho > c$.

It encloses both the conductors



(i.e) I & $-I$.

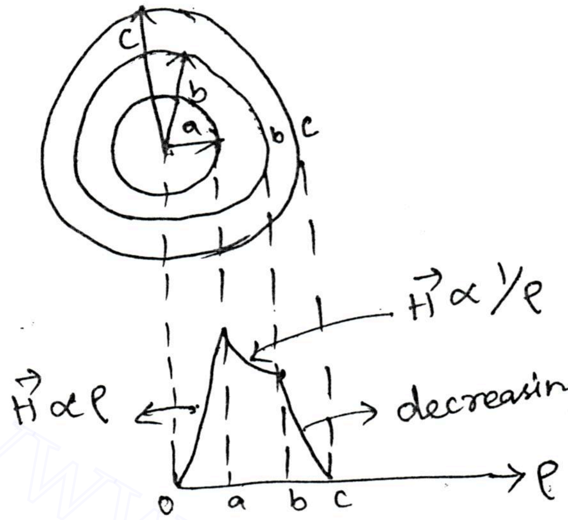
$$I_{enc} = I + (-I) = 0.$$

$$\int \vec{H}\cdot d\vec{l} = 0.$$

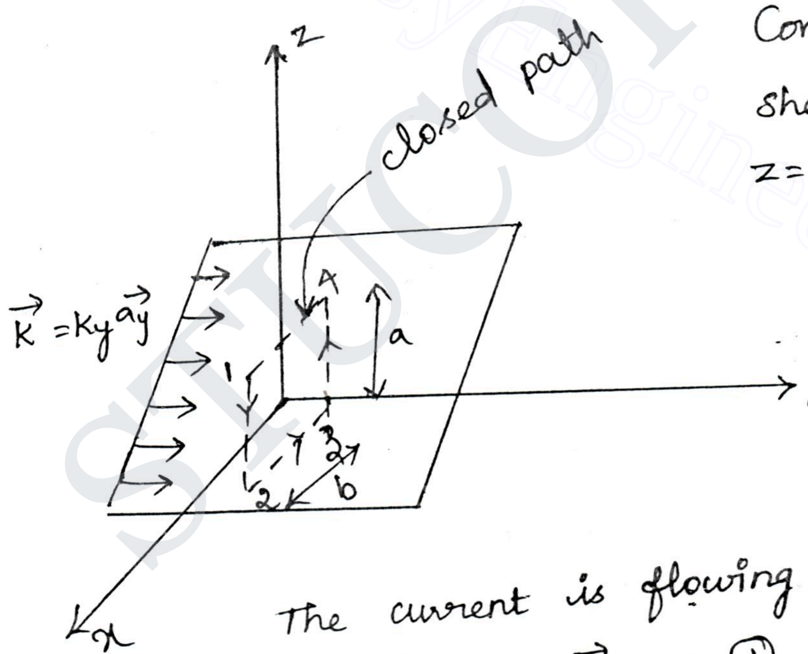
$$\boxed{\vec{H} = 0}$$

Magnetic field does not exist outside the cable.

Variation of $\vec{H}$ against ρ in coaxial cable.



3. $\vec{H}$ due to Infinite sheet of current:



Consider an Infinite sheet of current in the $z=0$ plane.

$K \to$ surface charge density.

The current is flowing in positive 'y' direction

hence $\vec{K} = K_y \vec{a_y} \to (1)$

Consider a closed path 12341 as shown in the figure. The width of the path is 'b' while the height is 'a'. It is perpendicular to the direction of current hence in xz plane.

The current flowing across the distance b is given by

$$I_{enc} = k_y b \longrightarrow (2)$$

Consider a magnetic lines of forces due to the current in $\vec{a_y}$ direction.

H_z components cancel each other. So $\vec{H}$ has the component only in x-direction.

$$\vec{H} = \begin{cases} H_x \vec{a_x} & \text{for } z>0 \\ -H_x \vec{a_x} & \text{for } z<0. \end{cases}$$

By Ampere's circuital law

$$\oint \vec{H}\cdot\vec{dl} = I_{enc}.$$

$$\int_1^2 \vec{H}\cdot\vec{dl} + \int_2^3 \vec{H}\cdot\vec{dl} + \int_3^4 \vec{H}\cdot\vec{dl} + \int_4^1 \vec{H}\cdot\vec{dl} = I_{enc} \longrightarrow (3)$$

For 1-2

$dl = dz\, \vec{a_z}$

$\vec{H}\cdot\vec{dl} = 0$

$(\because H_x \vec{a_x} \cdot dz \vec{a_z}$
$\vec{a_x}\cdot\vec{a_z} = 0)$

$\therefore \int_1^2 \vec{H}\cdot\vec{dl} = 0$

For 2-3 $(z<0)$

$dl = dx(\vec{a_x})$

$H = -H_x \vec{a_x}$

$$\int_2^3 \vec{H}\cdot\vec{dl} = \int_2^3 (-H_x \vec{a_x})\cdot dx(\vec{a_x})$$

$$= \int_2^3 H_x\, dx.$$

$$= H_x \int_0^b dx$$

$$\boxed{\int_2^3 \vec{H}\cdot\vec{dl} = H_x b}$$

for 3-4.

$$\int_3^4 \vec{H}\cdot\vec{dl} = \int_3^4 H_x\vec{a_x}\cdot dz\,\vec{a_z} \qquad [\vec{a_x}\cdot\vec{a_z}=0]$$

$$= 0.$$

for 4-1 $(z>0)$

$$\int_4^1 \vec{H}\cdot\vec{dl} = \int_4^1 (H_x\vec{a_x})\cdot(dx\,\vec{a_x})$$

$$= H_x\int_0^b dx$$

$$\boxed{\int_4^1 \vec{H}\cdot\vec{dl} = H_x b}$$

$$\therefore \oint \vec{H}\cdot dl = H_x b + H_x b.$$

$$\boxed{\oint \vec{H}\cdot dl = 2H_x b} \longrightarrow (4)$$

Therefore (3) will become

$$2H_x b = I_{enc}$$

$$2H_x \cancel{b} = K_y \cancel{b}.$$

$$\boxed{H_x = K_y/2}$$

$$\vec{H} = \begin{cases} K_y/2\,\vec{a_x} ; & z>0 \\ -K_y/2\,\vec{a_x} ; & z<0 \end{cases}$$

In general $\vec{H} = \frac{1}{2}(\vec{K}\times\vec{a_N})$

$\vec{a_N}$ - Unit Normal vector from the current sheet to the point at which $\vec{H}$ is to be obtained

SCALAR AND VECTOR MAGNETIC POTENTIALS.

V_m — scalar magnetic potential

$\vec{A}$ - Vector magnetic Potential

Vector identities used to define scalar and vector magnetic potentials

$$\nabla \times (\nabla V) = 0 \rightarrow (1)$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \rightarrow (2)$$

Scalar magnetic potential (V_m):

The scalar magnetic potential (V_m) is

$$V_m = -\int_a^b \vec{H} \cdot d\vec{l} \rightarrow (3)$$

If the V_m is scalar magnetic potential, it must satisfy eqn (1)

$$\nabla \times (\nabla V_m) = 0 \rightarrow (4)$$

V_m is related with $\vec{H}$ as

$$\vec{H} = -\nabla V_m \Rightarrow \nabla V_m = -\vec{H}$$

$$(4) \Rightarrow \nabla \times (-\vec{H}) = 0$$

$$\boxed{\nabla \times \vec{H} = 0} \rightarrow (5)$$

By point form of Amperé's circuital law,

$$\nabla \times \vec{H} = \vec{J} \rightarrow (6)$$

By comparing (5) & (6)

$$\boxed{\vec{J} = 0}$$

Scalar magnetic potential V_m can be defined for source free region where $\vec{J} = 0$.

$$\boxed{\text{i.e } \vec{H} = -\nabla V_m \text{ only for } \vec{J} = 0}$$

Laplace equation for scalar magnetic potential

WKT for a closed surface

$$\oint \vec{B} \cdot d\vec{s} = 0$$

By divergence Theorem.

$$\int (\nabla \cdot \vec{B}) dv = 0$$

$$\nabla \cdot \vec{B} = 0$$

Sub $\vec{B} = \mu \vec{H}$

$$\nabla \cdot (\mu \vec{H}) = 0$$

$$\nabla \cdot \vec{H} = 0.$$

Sub $\vec{H} = -\nabla V_m$

$$\nabla \cdot (-\nabla V_m) = 0$$

Laplace equation $\Rightarrow$ $\boxed{\nabla^2 V_m = 0}$ for $\vec{J} = 0$

Vector magnetic potential $(\vec{A})$:

If $\vec{A}$ is a vector magnetic potential, it must satisfy equation (2)

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \rightarrow (8)$$

WKT $\nabla \cdot \vec{B} = 0 \rightarrow (9)$

By comparing (8) & (9) $\boxed{\vec{B} = \nabla \times \vec{A}} \rightarrow (10)$

Curl of a vector magnetic potential is magnetic flux density.

By point form of Ampere's circuital law

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \times \frac{\vec{B}}{\mu} = \vec{J}$$

$$\nabla \times \vec{B} = \mu \vec{J}$$

sub $\vec{B} = \nabla \times \vec{A}$

$$\nabla \times (\nabla \times \vec{A}) = \mu \vec{J}$$

$$\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}$$

$$\vec{J} = \frac{1}{\mu}\left[\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right] \longrightarrow (11)$$

If $\vec{A}$ is known, $\vec{J}$ is obtained.

→ for $\vec{A}$, $\vec{J}$ is need not to be zero.

Poisson's Equation for magnetic Field:

for time varying field $\nabla \cdot \vec{A} = 0$

$$(11) \Rightarrow \vec{J} = \frac{1}{\mu}\left[-\nabla^2 A\right]$$

$$\boxed{\nabla^2 A = -\mu J} \longrightarrow (12)$$

$\vec{A}$ Due to Differential current Elements:

i) For differential element $d\vec{l}$ carrying current I,

the $\vec{A}$ is

$$\vec{A} = \oint_l \frac{\mu I d\vec{l}}{4\pi R} \text{ wb/m}$$

ii) For Surface $I\vec{dl} = \vec{K}ds$

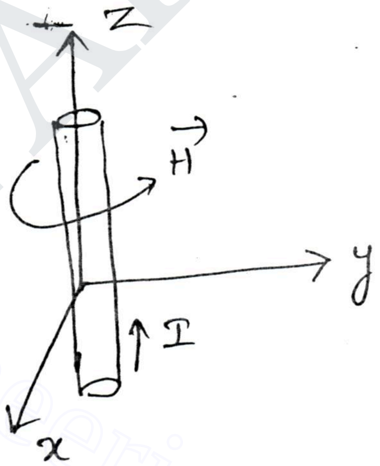
$$\vec{A} = \oint_S \frac{\mu \vec{K} ds}{4\pi R} \quad wb/m.$$

iii) For Volume $I\vec{dl} = \vec{J}dv$

$$\vec{A} = \oint_V \frac{\mu \vec{J} dv}{4\pi R} \quad wb/m.$$

Magnetic vector potential in the Region Surrounding an Infinitely Long straight Filamentary current I:

$\vec{H}$ due to an infinite long conductor is



$$\vec{H} = \frac{I}{2\pi\rho}\vec{a_\phi} \longrightarrow (1)$$

$$\vec{B} = \frac{\mu I}{2\pi\rho}\vec{a_\phi} \longrightarrow (2)$$

WKT $\vec{B} = \nabla \times \vec{A}$

$$\therefore \frac{\mu I}{2\pi\rho}\vec{a_\phi} = \frac{1}{\rho}\begin{vmatrix} \vec{a_\rho} & \rho\vec{a_\phi} & \vec{a_z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$

$$\frac{\mu I}{2\pi\rho}\vec{a_\phi} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\vec{a_\rho} + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right]\vec{a_\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\vec{a_z}$$

By equating the coefficients of $\vec{a_\phi}$

$$\frac{\mu I}{2\pi\rho} = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho} \longrightarrow (3)$$

$\vec{B}$ is a function of ρ only.

so $\vec{A}$ is also change with ρ only.

$$\frac{\partial A_\rho}{\partial z} = 0.$$

$$③ \Rightarrow \quad -\frac{\partial A_z}{\partial \rho} = \frac{\mu I}{2\pi\rho}$$

$$\frac{\partial A_z}{\partial \rho} = -\frac{\mu I}{2\pi\rho}$$

$\int$ing on both sides.

$$A_z = -\frac{\mu I}{2\pi} \ln \rho + C_1 \longrightarrow ④$$

at $\rho = \rho_0$, $A_z = 0$

$$④ \Rightarrow 0 = -\frac{\mu I}{2\pi} \ln \rho_0 + C_1$$

$$\boxed{C_1 = \frac{\mu I}{2\pi} \ln \rho_0} \longrightarrow ⑤$$

$\therefore$ sub ⑤ in ④

$$A_z = -\frac{\mu I}{2\pi} \ln(\rho) + \frac{\mu I}{2\pi} \ln(\rho_0)$$

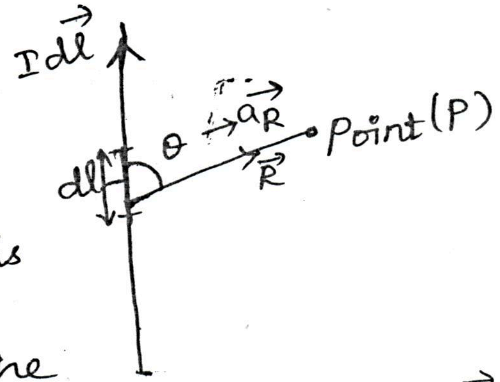
$$A_z = \frac{\mu I}{2\pi} \ln\left(\frac{\rho_0}{\rho}\right).$$

$$\vec{A} = A_z \vec{a_z}$$

$$\boxed{\vec{A} = \frac{\mu I}{2\pi} \ln\left(\frac{\rho_0}{\rho}\right) \vec{a_z}} \quad \text{wb/m}$$

BIOT SAVART LAW AND APPLICATIONS :

(i) BIOT SAVART LAW.



Biot Savart law states that, the differential magnetic field $\vec{dH}$ of a conductor at point p is

(a) directly proportional to the product of current (I) and differential length ($\vec{dl}$).

(b) directly proportional to the sine of angle between the conductor and the line joing pt p and $\vec{dl}$.

(c) inversely proportional to the square of the distance between p and dl.

$$\vec{dH} \propto \frac{I\vec{dl}\sin\theta}{R^2}$$

$$\vec{dH} = \frac{k\, I\vec{dl}\sin\theta}{R^2}$$; where k - Proportionality const. $k = 1/4\pi$

$$\vec{dH} = \frac{I\vec{dl}\sin\theta}{4\pi R^2}$$

$$\vec{dH} = \frac{I\vec{dl}\times\vec{a_R}}{4\pi R^2} \qquad \vec{a_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}.$$

$$\vec{dH} = \frac{I\vec{dl}\times\vec{R}}{4\pi R^3}.$$

$$\vec{H} = \int_l \frac{I\vec{dl}\times\vec{R}}{4\pi R^3}$$

Biot savart law interms of distributed sources:

(i) If ds is the differential surface area considered of a surface having current density $\vec{K}$

$$I\vec{dl} = \vec{K}\,ds$$

$$\therefore \vec{H} = \int_S \frac{\vec{K} \times \vec{a_R}}{4\pi R^2}\,ds \quad A/m.$$

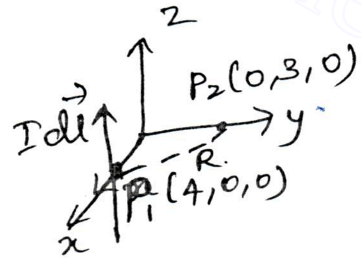
(ii) If the current density in a volume of a given conductor is $\vec{J}$ measured in A/m^2, then for a differential volume dv

$$I\vec{dl} = \vec{J}\,dv$$

$$\vec{H} = \int_V \frac{\vec{J} \times \vec{a_R}}{4\pi R^2}\,dv \quad A/m.$$

Problem:

1. Find the incremental field strength at P_2 due to current element of $2\pi \vec{a_z}$ μAm at P_1. The coordinates of P_1 & P_2 are $(4,0,0)$ & $(0,3,0)$



Given: $I\vec{dl} = 2\pi \vec{a_z}\ \mu Am.$

$$= (2\pi \times 10^{-6})\,\vec{a_z}\ Am.$$

soln: $\vec{R} = P_2 - P_1$

$$\vec{R} = (0-4)\vec{a_x} + (3-0)\vec{a_y} + (0-0)\vec{a_z}$$

$$\vec{R} = -4\vec{a_x} + 3\vec{a_y}$$

$$R = |\vec{R}| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$$

$$I\vec{dl} \times \vec{R} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ 0 & 0 & (2\pi \times 10^{-6}) \\ -4 & 3 & 0 \end{vmatrix}$$

$$I\vec{dl} \times \vec{R} = \vec{a_x}(-6\pi \times 10^{-6}) - \vec{a_y}(+8\pi \times 10^{-6}) + \vec{a_z}(0)$$

$$= -2\pi \times 10^{-6}\left[3\vec{a_x} + 4\vec{a_y}\right]$$

By Biot Savart law

$$\vec{dH} = \frac{I\vec{dl} \times \vec{a_R}}{4\pi R^2}$$

$$= \frac{I\vec{dl} \times \vec{R}}{4\pi R^3}$$

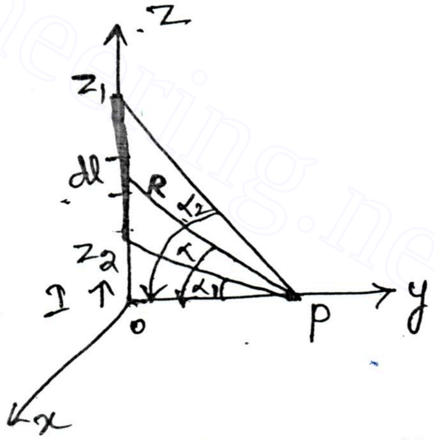
$$= \frac{2\pi \times 10^{-6}\left[3\vec{a_x} + 4\vec{a_y}\right]}{4\pi (5)^3}$$

$$\boxed{\vec{dH} = -12\vec{a_x} - 16\vec{a_y} \ n\,A/m}$$

Applications of Biot Savart law:

1. $\vec{H}$ due to Finite and Infinite line

→ Consider a conductor of finite length placed along z-axis, which carries current I.



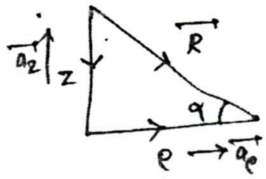
→ Perpendicular distance of point P from z axis is R.

– The conductor is placed such that its one end is at $z = z_1$ while other at $z = z_2$.

– Consider a differential length $\vec{dl}$ along z-axis at a distance z from origin.

$$\vec{dl} = dz\,\vec{a_z}$$

$$\vec{R} = \rho\,\vec{a_\rho} - z\,\vec{a_z}$$



$$R = |\vec{R}| = \sqrt{\rho^2 + z^2}$$

$$\vec{dl} \times \vec{R} = \begin{vmatrix} \vec{a_\rho} & \vec{a_\phi} & \vec{a_z} \\ 0 & 0 & dz \\ \rho & 0 & -z \end{vmatrix} = \vec{a_\phi}\,(\rho\,dz).$$

By Biot Savart law,

$$\vec{dH} = \frac{I\,\vec{dl} \times \vec{a_R}}{4\pi R^2} = \frac{I\,\vec{dl} \times \vec{R}}{4\pi R^3} \quad \text{—— (1)}$$

$$\vec{dH} = \frac{I\,\rho\,dz\,\vec{a_\phi}}{4\pi(\rho^2 + z^2)^{3/2}}$$

$$\vec{H} = \iint_S \frac{I\,\rho\,dz\,\vec{a_\phi}}{4\pi(\rho^2 + z^2)^{3/2}} \quad \text{—— (2)}$$

from the figure

$$\tan\alpha = \frac{z}{\rho}$$

$$\boxed{z = \rho\tan\alpha}$$

$$\boxed{dz = \rho\sec^2\alpha\, d\alpha}$$

$$(2) \Rightarrow \vec{H} = \frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2\sec^2\alpha\,d\alpha}{(\rho^2 + \rho^2\tan^2\alpha)^{3/2}}\,\vec{a_\phi}$$

$$= \frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\cancel{\rho^2}\,\cancel{\sec^2}\alpha}{\rho^{\cancel{3}}\sec^{\cancel{3}}\alpha}\,d\alpha\,\vec{a_\phi}$$

$$= \frac{I}{4\pi e} \int_{\alpha_1}^{\alpha_2} \cos\alpha \, d\alpha \, \vec{a_\phi}$$

$$= \frac{I}{4\pi e} \left[\sin\alpha \right]_{\alpha_1}^{\alpha_2} \vec{a_\phi}$$

$$\boxed{\vec{H} = \frac{I}{4\pi e} \left[\sin\alpha_2 - \sin\alpha_1 \right] \vec{a_\phi}} \quad A/m.$$

$$\vec{B} = \mu \vec{H}$$

$$\boxed{\vec{B} = \frac{\mu I}{4\pi e} \left[\sin\alpha_2 - \sin\alpha_1 \right] \vec{a_\phi}} \quad Wb/m^2.$$

for infinite line,

$$\alpha_1 = -\frac{\pi}{2} \quad \& \quad \alpha_2 = \frac{\pi}{2}$$

$$\therefore \vec{H} = \frac{I}{4\pi e} \left[\sin\frac{\pi}{2} - \sin\left(\frac{-\pi}{2}\right) \right] \vec{a_\phi}$$

$$= \frac{I}{4\pi e} \left[1 + 1 \right] \vec{a_\phi}$$

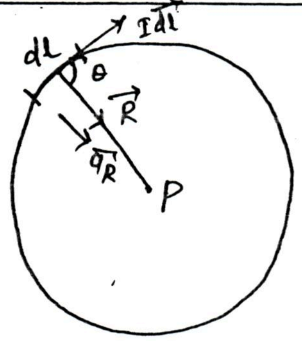
$$= \frac{I}{{}_2\cancel{4}\pi e} \times \cancel{2} \, \vec{a_\phi}$$

$$\boxed{\vec{H} = \frac{I}{2\pi e} \vec{a_\phi}} \quad A/m$$

$$\vec{B} = \mu \vec{H}$$

$$\boxed{\vec{B} = \frac{\mu I}{2\pi e} \vec{a_\phi}} \quad Wb/m^2.$$

$\vec{H}$ AT THE CENTER OF A CIRCULAR CONDUCTOR:



- Consider a current carrying Conductor arranged in a Circular form.
- The Conductor carries a direct current I.
- Consider a differential length dl.
- The $\vec{H}$ at point P is to be obtained.

θ – Angle between $I\vec{dl}$ and $\vec{a_R}$ $= 90°$

$\vec{R}$ – distance vector.

By Biot Savart law,

$$d\vec{H} = \frac{I\vec{dl} \times \vec{R}}{4\pi R^3} = \frac{I\vec{dl} \times \vec{a_R}}{4\pi R^2}$$

$$d\vec{H} = \frac{I\, dl \sin\theta\, \vec{a_N}}{4\pi R^2}$$

$$\vec{H} = \frac{I \sin\theta\, \vec{a_N}}{4\pi R^2} \oint dl$$

$$= \frac{I \sin\theta\, \vec{a_N}}{4\pi R^2} \times 2\pi R$$

$$= \frac{I \sin\theta}{2R} \vec{a_N}$$

here $\theta = 90°$

$$\boxed{\vec{H} = \frac{I}{2R} \vec{a_N}}$$

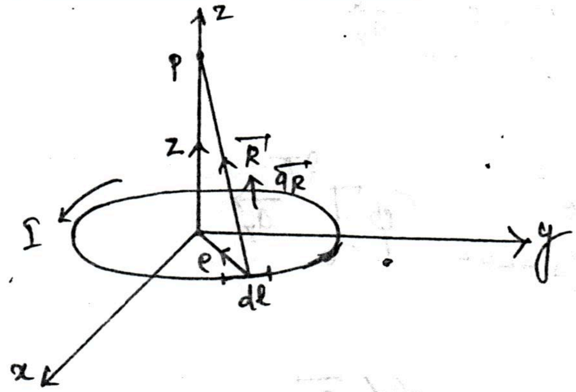
If the circular loop is placed in xy plane,

$$\vec{a_N} = \vec{a_z}$$

$$\therefore \boxed{\vec{H} = \frac{I}{2R} \vec{a_z}} \text{ A/m.}$$

$$\boxed{\vec{B} = \frac{\mu I}{2R} \vec{a_z}} \text{ Wb/m}^2$$

$\vec{H}$ ON THE AXIS OF CIRCULAR LOOP:

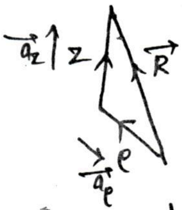


- Consider a circular loop carrying a current I placed in xy plane with z-axis as the axis.
- The magnetic field intensity $\vec{H}$ at point P is to be obtained.
- Consider a differential length $\vec{dl}$ on the circular loop.
- The radius of the circular loop is ρ.
- The point P is at a distance of z from the plane of the circular loop.

By Biot Savart law,

$$d\vec{H} = \frac{I\,\vec{dl} \times \vec{a_R}}{4\pi R^2} = \frac{I\,\vec{dl} \times \vec{R}}{4\pi R^3} \quad \text{---} \; (1)$$

here $\vec{dl} = \rho\, d\phi\, \vec{a_\phi}$



$$\vec{R} = -\rho\,\vec{a_\rho} + z\,\vec{a_z}$$

$$R = |\vec{R}| = \sqrt{\rho^2 + z^2}.$$

$$\vec{dl} \times \vec{R} = \begin{vmatrix} \vec{a_\rho} & \vec{a_\phi} & \vec{a_z} \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & z \end{vmatrix} = \vec{a_\rho}\,(z\rho\, d\phi) + \vec{a_z}\,(\rho^2 d\phi)$$

$$\vec{dl} \times \vec{R} = \rho z\, d\phi\, \vec{a_\rho} + \rho^2 d\phi\, \vec{a_z}$$

$$(1) \Rightarrow \quad d\vec{H} = \frac{I\,(\rho z\, d\phi\, \vec{a_\rho} + \rho^2 d\phi\, \vec{a_z})}{4\pi(\rho^2 + z^2)^{3/2}}$$

$$\vec{H} = \frac{I}{4\pi}\left[\int_0^{2\pi} \frac{\rho z\, d\phi}{(\rho^2+z^2)^{3/2}}\,\vec{a_\rho} + \int_0^{2\pi} \frac{\rho^2\, d\phi}{(\rho^2+z^2)^{3/2}}\,\vec{a_z}\right]$$

Due to radial symmetry, there is no $\vec{a_\rho}$ component.

$$\vec{H} = \frac{I}{4\pi}\int_0^{2\pi} \frac{\rho^2\, d\phi}{(\rho^2+z^2)^{3/2}}\, \vec{a_z}$$

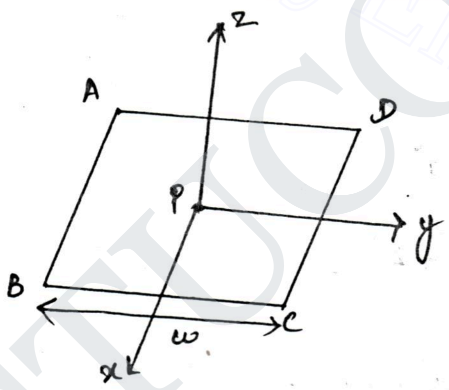
$$= \frac{I}{4\pi} \times \frac{\rho^2}{(\rho^2+z^2)^{3/2}} \left[\phi\right]_0^{2\pi} \vec{a_z}$$

$$= \frac{I}{2\,\cancel{4\pi}} \times \frac{\rho^2}{(\rho^2+z^2)^{3/2}} \times \cancel{2\pi}\, \vec{a_z}$$

$$\boxed{\vec{H} = \frac{I\rho^2}{2(\rho^2+z^2)^{3/2}}\, \vec{a_z}} \quad A/m.$$

$$\boxed{\vec{B} = \frac{\mu I \rho^2}{2(\rho^2+z^2)^{3/2}}\, \vec{a_z}} \quad Wb/m^2$$

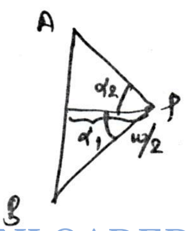
$\vec{H}$ ON THE CENTER OF SQUARE LOOP:



- Consider a square loop with side 'w' Carrying a Current I lies in xy plane.
- Let P be the point where the $\vec{H}$ is to be determined.
- The field intensity at a distance 'h' from any current carrying Conductor is

$$\vec{H} = \frac{I}{4\pi h}\left[\sin\alpha_2 - \sin\alpha_1\right]\vec{a_z}.$$

for line AB:



$$\vec{H}_{AB} = \frac{I}{4\pi\left(\frac{w}{2}\right)}\left[\sin 45^\circ - \sin(-45^\circ)\right]\vec{a_z}$$

$$= \frac{I}{2\pi w}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right]\vec{a_z}$$

$$\vec{H} = \frac{I}{\pi W \sqrt{2}} \vec{a_z}$$

The Field Intensity due to sides AB, BC, CD and DA are equal

$$\vec{H} = 4 \vec{H}_{AB}$$

$$= 4 \times \frac{I}{\pi W \sqrt{2}} \vec{a_z}$$

$$\boxed{\vec{H} = \frac{2\sqrt{2} I}{\pi W} \vec{a_z}} \text{ A/m.}$$

$$\boxed{\vec{B} = \frac{\mu I 2\sqrt{2}}{\pi W} \vec{a_z}} \text{ Wb/m}^2$$

Problem:

1. A circular loop located on $x^2+y^2=9, z=0$ carries a direct current of 10A along a_ϕ. Determine $\vec{H}$ at $(0,0,4)$ and $(0,0,-4)$

Given: $I = 10A$; $x^2+y^2 = 9$.

Solution:

$$\vec{H} = \frac{I\rho^2}{2(\rho^2+z^2)^{3/2}} \vec{a_z}$$

In cylindrical co-ordinate system.

$$\rho = \sqrt{x^2+y^2} = \sqrt{9}$$

$$\boxed{\rho = 3}$$

i) $\vec{H}$ at $(0,0,4)$ [Here $z=4$]

$$\therefore \vec{H} = \frac{10 \times (3)^2}{2(9+16)^{3/2}} \vec{a_z}$$

$$\boxed{\vec{H} = 0.36 \vec{a_z}}$$

ii) $\vec{H}$ at $(0,0,-4)$ $(z=-4)$

$$\vec{H} = \frac{10 \times (3)^2}{2(3^2+(-4)^2)^{3/2}} \vec{a_z}$$

$$\boxed{\vec{H} = 0.36 \vec{a_z}}$$

MAGNETIC BOUNDARY CONDITIONS INVOLVING MAGNETIC FIELDS :

— The Conditions existing at the boundary of two medium when the field passes from one medium to other are Called as boundary Conditions.

— To analyze the boundary Conditions the following equations are required.

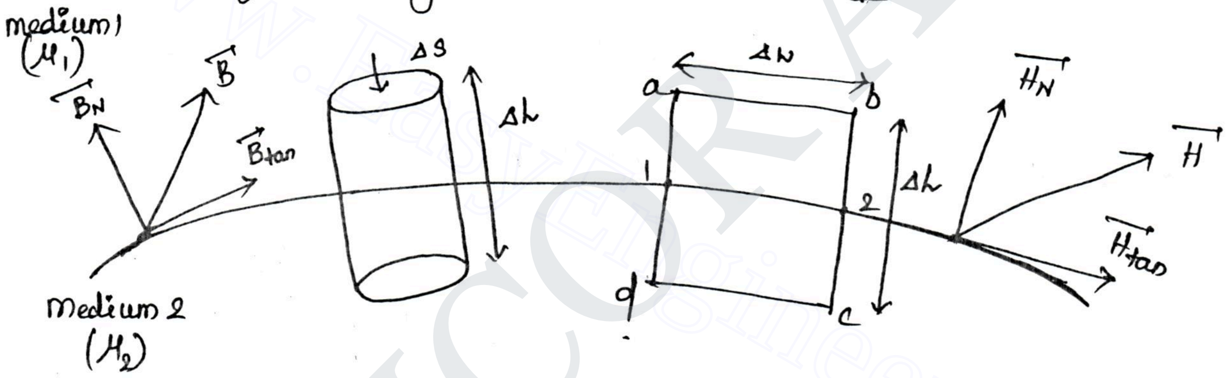
(i) $\int \vec{H} \cdot \vec{dl} = I$

(ii) $\int \vec{B} \cdot \vec{dS} = 0$

— The field Intensity ($\vec{H}$) and Flux density ($\vec{B}$) is required to be decomposed into tangential Components and normal components.

— Consider a boundary between two isotrophic, homogeneous linear materials with different permeabilities μ_1 and μ_2.

— To determine the boundary Conditions the closed path (abcda) and gaussian surface as cylinder are used.



Tangential Components at the boundary :

By Ampere's Circuital law,

$$\int \vec{H} \cdot \vec{dl} = I \quad \text{——— (1)}$$

for the closed path abcda, equation (1) can be written as,

$$\int_a^b \vec{H} \cdot \vec{dl} + \int_b^c \vec{H} \cdot \vec{dl} + \int_c^d \vec{H} \cdot \vec{dl} + \int_d^a \vec{H} \cdot \vec{dl} = I \quad \text{——— (2)}$$

The sides a-b and cd are parallel to tangential direction to the surface while other two are normal to the surface at the boundary.

for the line ab. $\vec{H}$ is assumed to be Constant (i,e) $\vec{H} = H_{tan}$,

$$\int_a^b \vec{H} \cdot \cancel{\vec{dl}} = H_{tan} \int_a^b \vec{dl} = H_{tan} \Delta w \quad \text{——— (A)}$$

The line cd is opposite to the direction of ab.

$$\int_c^d \vec{H} \cdot \vec{dl} = -H_{tan_2} \int_c^d \vec{dl} = -H_{tan_2} \Delta w \quad \text{——— (B)}$$

for the line b-c

$$\int_b^c \overline{H'}\cdot\overline{dl} = \int_b^2 \overline{H'}\cdot\overline{dl} + \int_2^c \overline{H'}\cdot\overline{dl}$$

for the line b-2, $\overline{H'} = H_{N_1}$

for the line 2-c, $\overline{H'} = H_{N_2}$

$$\int_b^c \overline{H'}\cdot\overline{dl} = H_{N_1}\int_b^2 \overline{H'}\cdot\overline{dl} + H_{N_2}\int_2^c \overline{H'}\cdot\overline{dl}$$

$$= H_{N_1}\left(\frac{\Delta h}{2}\right) + H_{N_2}\left(\frac{\Delta h}{2}\right) \quad \text{——— (C)}$$

The line da is opposite to the direction of b-c.

$$\therefore \int_d^a \overline{H'}\cdot\overline{dl} = -H_{N_1}\left(\frac{\Delta h}{2}\right) - H_{N_2}\left(\frac{\Delta h}{2}\right) \quad \text{——— (D)}$$

Sub (A), (B), (C) & (D) in (2)

$$(2) \Rightarrow H_{tan_1}\Delta w + \cancel{H_{N_1}\left(\frac{\Delta h}{2}\right)} + \cancel{H_{N_2}\left(\frac{\Delta h}{2}\right)} - H_{tan_2}\Delta w - \cancel{H_{N_1}\left(\frac{\Delta h}{2}\right)} - \cancel{H_{N_2}\left(\frac{\Delta h}{2}\right)} = I$$

$$(H_{tan_1} - H_{tan_2})\cancel{\Delta w} = K\cancel{\Delta w}$$

$$\boxed{H_{tan_1} - H_{tan_2} = K} \quad \text{——— (3)}$$

In vector form

$$\overline{H_{tan_1}} - \overline{H_{tan_2}} = \overline{a_{N_{12}}} \times \overline{K}$$

$$(3) \Rightarrow \boxed{\frac{B_{tan_1}}{\mu_1} - \frac{B_{tan_2}}{\mu_2} = K} \quad \text{——— (4)}$$

If the medium is not a conductor, $k = 0$.

$$(3) \Rightarrow H_{tan_1} - H_{tan_2} = 0$$

$$\boxed{H_{tan_1} = H_{tan_2}} \quad \text{——— (5)}$$

$$(4) \Rightarrow \frac{B_{tan_1}}{\mu_1} - \frac{B_{tan_2}}{\mu_2} = 0$$

$$\frac{B_{tan_1}}{\mu_1} = \frac{B_{tan_2}}{\mu_2} \Rightarrow \frac{B_{tan_1}}{B_{tan_2}} = \frac{\mu_1}{\mu_2}$$

$$\frac{B_{tan_1}}{B_{tan_2}} = \frac{\cancel{\mu_0}\mu_{r_1}}{\cancel{\mu_0}\mu_{r_2}}$$

$$\boxed{\frac{B_{tan_1}}{B_{tan_2}} = \frac{\mu_{r_1}}{\mu_{r_2}}} \quad \text{——— (6)}$$

Normal Components at the boundary:

WKT

$$\int \vec{B}\cdot d\vec{s} = 0 \quad \text{——— (7)}$$

for the gaussian surface equation (7) can be written as

$$(7) \Rightarrow \int_{top} \vec{B}\cdot d\vec{s} + \int_{bottom} \vec{B}\cdot d\vec{s} + \int_{lateral} \vec{B}\cdot d\vec{s} = 0 \quad \text{——— (8)}$$

As $\Delta h \to 0$, $\int_{lateral} \vec{B}\cdot d\vec{s} = 0$ ——— (E)

for top surface $\vec{B}$ is assumed to be constant (i.e) $\vec{B} = B_{N_1}$

$$\therefore \int_{top} \vec{B}\cdot d\vec{s} = B_{N_1}\int_{top} d\vec{s}$$

$$= B_{N_1}\,\Delta S \quad \text{——— (F)}$$

for bottom surface $\vec{B}$ is assumed to be constant (i.e) $\vec{B} = -B_{N_2}$

$$\therefore \int_{bottom} \vec{B}\cdot d\vec{s} = -B_{N_2}\int_{bottom} d\vec{s}$$

$$= -B_{N_2}\,\Delta S \quad \text{——— (G)}$$

Sub eqn (E), (F) and (G) in eqn (8)

$$(8) \Rightarrow B_{N_1}\Delta S - B_{N_2}\Delta S = 0$$

$$(B_{N_1} - B_{N_2})\Delta S = 0$$

$$B_{N_1} - B_{N_2} = 0$$

$$\boxed{B_{N_1} = B_{N_2}} \quad \text{——— (9)}$$

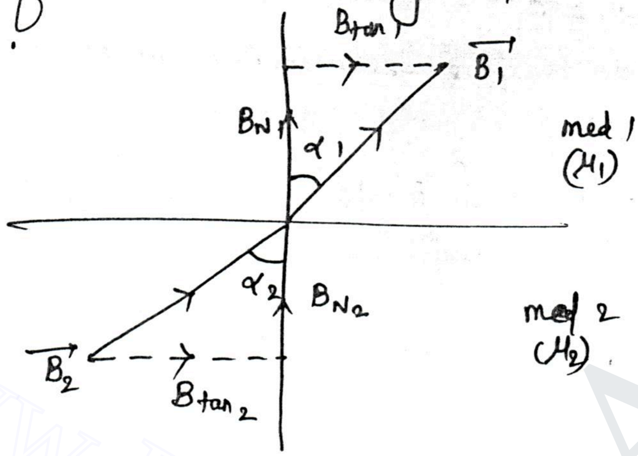
Equation (9) can be written as,

$$\mu_1 H_{N_1} = \mu_2 H_{N_2}$$

$$\frac{H_{N_1}}{H_{N_2}} = \frac{\mu_2}{\mu_1} = \frac{\cancel{\mu_0}\mu_{r_2}}{\cancel{\mu_0}\mu_{r_1}}$$

$$\boxed{\frac{H_{N_1}}{H_{N_2}} = \frac{\mu_{r_2}}{\mu_{r_1}}}$$

- Let the fields makes an angle α_1 and α_2 with the normal to the interface.



from the figure

$$\tan\alpha_1 = \frac{B_{tan\,1}}{B_{N_1}}$$

$$\tan\alpha_2 = \frac{B_{tan\,2}}{B_{N_2}}$$

$$\Rightarrow \frac{\tan\alpha_1}{\tan\alpha_2} = \frac{B_{tan\,1}}{B_{N_1}} \times \frac{B_{N_2}}{B_{tan\,2}}$$

WKT $B_{N_1} = B_{N_2}$

$$= \frac{B_{tan\,1}}{B_{tan\,2}}$$

$$\boxed{\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\mu_{r_1}}{\mu_{r_2}}}$$

THE MAGNETIC CIRCUIT:

- "The path traced by a magnetic flux is called magnetic circuit"
- Magnetic circuits are analogous to electric circuit.
- If the analogy between electric and magnetic circuit is known, the techniques for the analysis of magnetic circuits are simple.
- Examples for magnetic circuits are Transformers, Toroids, Motors, Generators, Relays and magnetic recording devices.
- Single magnetic line of flux or all parallel magnetic lines of flux may be considered as magnetic circuit.

COMPARISON BETWEEN ELECTRIC CIRCUIT AND MAGNETIC CIRCUIT:

Electric Circuit	Magnetic Circuit
1. Path traced by the current is called electric circuit.	Path traced by the magnetic flux is called magnetic circuit.
2. Voltage source is the part of the closed path.	The current carrying coil will surround the magnetic circuit.
3. electro motive force (emf) is the driving force.	magneto motive force (mmf) is the driving force.
4. The electromotive force (V) is $V = \int_a^b \vec{E}\cdot d\vec{l}$.	The magneto motive force (V_m) is $V_m = \int_a^b \vec{H}\cdot d\vec{l}$.
5. Relation between $\vec{E}$ and V is $\vec{E} = -\nabla V$	Relation between $\vec{H}$ and V_m is $\vec{H} = -\nabla V_m$
6. The ohm's law for electric circuit is $V = IR$, $\vec{J} = \sigma \vec{E}$	The ohm's law for magnetic circuit is $V_m = \phi \mathcal{R}$, $\vec{B} = \mu \vec{H}$.
7. The total current in electric circuit is $I = \int_S \vec{J}\cdot d\vec{S}$	The total flux in magnetic circuit is $\phi = \int_S \vec{B}\cdot d\vec{S}$
8. Resistance (R) is the ratio of emf to current $R = \frac{V}{I}$	Reluctance ($\mathcal{R}$) is the ratio of mmf to flux $\mathcal{R} = \frac{V_m}{\phi}$

9. The resistance (R) for the length d is; $R = \frac{d}{\sigma S}$	The reluctance ($\mathcal{R}$) for the length d is $\mathcal{R} = \frac{d}{\mu S}$
10. Conductance for electric circuit is $G = \frac{1}{R} = \frac{\sigma S}{d}$	Permeance for magnetic circuit is $P = \frac{1}{\mathcal{R}} = \frac{\mu S}{d}$
11. For electric circuit, the closed line integral of $\vec{E}$ is zero. $\oint \vec{E} \cdot \vec{dl} = 0$	For magnetic circuit, the closed line integral of $\vec{H}$ is equal to total current. $\oint \vec{H} \cdot \vec{dl} = I$ for N - turns coil $\oint \vec{H} \cdot \vec{dl} = NI$.
12. Kirchoff's law for electric circuit are $\sum I = 0$ $\sum emf = 0$	Kirchoff's law for magnetic circuits are $\sum \phi = 0$ $\sum mmf = \sum \phi \mathcal{R}$.

PROBLEM.

A magnetic circuit employs an air core toroid with 500 turns, cross sectional area 6 cm², mean radius 15 cm and 4 A coil current. Determine the reluctance of the circuit, flux density (B) and field Intensity (H).

Given:

$N = 500$

$S = 6 \text{ cm}^2 = 6 \times 10^{-4} \text{ m}^2$

$r = 15 \times 10^{-2} \text{ m}$

$I = 4 \text{ A}$.

Solution:

length of the toroid $\}$ $d =$ Circumference of the toroid

$= 2\pi r$

$$\boxed{d = 2\pi \times 15 \times 10^{-2}} \text{ m}$$

(i) Reluctance, $\mathcal{R} = \frac{d}{\mu S} = \frac{d}{\mu_0 \mu_r S}$

$$\mathcal{R} = \frac{2\pi \times 15 \times 10^{-2}}{4\pi \times 10^{-7} \times 1 \times 6 \times 10^{-4}}$$

$$\boxed{\mathcal{R} = 1.25 \times 10^{9} \ A.t/Wb}$$

(ii) Flux density, $B = \frac{\phi}{S}$

where $\phi = \frac{V_m}{\mathcal{R}} = \frac{NI}{\mathcal{R}} = \frac{500 \times 4}{1.25 \times 10^{9}} = 1.6 \times 10^{-6}$ Wb

$$B = \frac{1.6 \times 10^{-6}}{6 \times 10^{-4}}$$

$$\boxed{B = 2.667 \times 10^{-3} \ Wb/m^2}$$

(iii) Field Intensity, $H = \frac{B}{\mu}$

$$H = \frac{2.667 \times 10^{-3}}{4\pi \times 10^{-7}}$$

$$\boxed{H = 2122.1 \ A.t/m}$$

POTENTIAL ENERGY AND FORCES ON MAGNETIC MATERIALS

The Potential energy stored in magnetic field is

$$W_m = \frac{1}{2} \int_V \bar{B} \cdot \bar{H} \, dV \quad \text{——— (1)}$$

By $\bar{B} = \mu \bar{H}$

(1) $\Rightarrow$
$$W_m = \frac{1}{2} \int \mu H^2 \, dV \quad \text{——— (2)}$$

By $\bar{H} = \frac{\bar{B}}{\mu}$

(2) $\Rightarrow$
$$W_m = \frac{1}{2} \int \frac{B^2}{\mu} \, dV \quad \text{——— (3)}$$

Sub $B = \mu H$

(41)

$$\Delta L = \frac{\mu H \Delta x \Delta Z}{\Delta I} \quad \text{——— (4)}$$

The current flowing through the Conducting sheets present at the top and bottom is in y-direction.

$$\therefore \Delta I = H \Delta y \quad \text{——— (5)}$$

The energy stored in inductor of differential volume is

$$\Delta W_m = \frac{1}{2} (\Delta L)(\Delta I)^2 \quad \text{——— (6)}$$

Sub (4) and (5) in (6)

$$\Delta W_m = \frac{1}{2}\left[\frac{\mu H \Delta x \Delta Z}{\cancel{H \Delta y}}\right](H \Delta y)^{\cancel{2}}$$

$$= \frac{1}{2} \mu H^2 \Delta x \Delta y \Delta Z$$

$$\Delta W_m = \frac{1}{2} \mu H^2 \Delta V \quad \text{——— (7)}$$

The magnetostatic energy density function is

$$w_m = \lim_{\Delta V \to 0} \frac{\Delta W_m}{\Delta V}$$

$$= \lim_{\Delta V \to 0} \frac{\frac{1}{2} \mu H^2 \cancel{\Delta V}}{\cancel{\Delta V}}$$

$$\boxed{w_m = \frac{1}{2} \mu H^2} \quad J/m^3. \quad \text{——— (8)}$$

(8) $\Rightarrow$ $$\boxed{w_m = \frac{1}{2} \frac{B^2}{\mu}} \quad \text{——— (9)} \qquad \because H = \frac{B}{\mu}.$$

(8) $\Rightarrow$ $$w_m = \frac{1}{2} (\mu H) H$$

$$\boxed{w_m = \frac{1}{2} B H} \quad \text{——— (10)}$$

For linear medium, the energy in the magnetic field is

$$W_m = \int w_m \, dV$$

 From equation (3), the rate of increase is

$$dW_m = \frac{1}{2}\frac{B_{st}^2}{\mu_0} S \cdot dl \quad \text{——— (4)}$$

The rate of increase in W_m interms of force can be expressed as

$$dW_m = F\,dl \quad \text{——— (5)}$$

By comparing (4) & (5)

The force on a magnetic material is

$$\boxed{F = \frac{1}{2}\frac{B_{st}^2}{\mu_0} S} \quad \text{——— (6)}$$

The tractive pressure can be expressed as

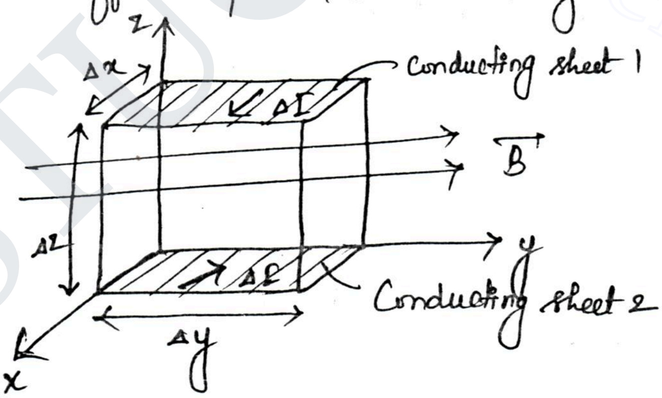
$$(6) \Rightarrow \quad P = \frac{F}{S} = \frac{B_{st}^2}{2\mu_0} = \frac{1}{2} B_{st} H_{st} = \frac{1}{2}\mu_0 H_{st}^2 .$$

ENERGY STORED IN MAGNETIC FIELDS : (MAGNETIC ENERGY)

The energy stored in inductor is

$$W_m = \frac{1}{2} L I^2 \quad \text{——— (1)}$$

– Consider a differential volume in magnetic field $\vec{B}$.



– Consider the top and bottom surfaces of differential volume as conducting sheets with current ΔI.

The inductance of the differential volume is

$$\Delta L = \frac{\Delta\phi}{\Delta I} = \frac{B\,\Delta S}{\Delta I} \quad \text{——— (2)}$$

where ΔS = Differential surface area $= \Delta x\,\Delta z$

$$(2) \Rightarrow \quad \Delta L = \frac{B\,\Delta x\,\Delta z}{\Delta I} \quad \text{——— (3)}$$

$$\boxed{W_m = \frac{1}{2}\int \vec{B}\cdot\vec{H}^T\, dV} \quad \text{——} \ (11)$$

(or)

$$\boxed{W_m = \frac{1}{2}\int \mu H^2\, dV} \quad \text{——} \ (12)$$

(or)

$$\boxed{W_m = \frac{1}{2}\int \frac{B^2}{\mu}\, dV} \quad \text{——} \ (13)$$

PROBLEM.

A current of 2A is flowing in an inductance of 100 mH. What is the energy stored in the inductor?

Given: $I = 2A$

$L = 100\, mH$

Solution:

$$W_m = \frac{1}{2} L I^2$$

$$= \frac{1}{2} \times 100 \times 10^{-3} \times (2)^2$$

$$= 200 \times 10^{-3}\ J$$

$$\boxed{W_m = 0.2\ J}$$

INDUCTANCE:

- Inductance is the energy storing element. Here the energy is stored in the form of magnetic field.
- Any current carrying conductor produces a magnetic field around it.
- If there are N-turns the flux lines link the current N-times. Therefore the total flux linkage is $\lambda = N\phi$.
- If I changes with time, λ also changes with time.

$$\left.\begin{matrix}\text{emf}\\ \text{induced}\end{matrix}\right\} = -\frac{d\lambda}{dt} = -\frac{d}{dt}(N\phi)$$

$$= -N\frac{d\phi}{dt}.$$

$$= -N\left[\frac{d\phi}{dI} \times \frac{dI}{dt}\right]$$

$$\left.\begin{matrix}\text{emf}\\ \text{induced}\end{matrix}\right\} = -L\frac{dI}{dt}$$

where $L = N\frac{d\phi}{dI}$.

- The inductance is classified into two types
  - (i) Self Inductance / Inductance
  - (ii) Mutual Inductance.

SELF INDUCTANCE / INDUCTANCE (L):

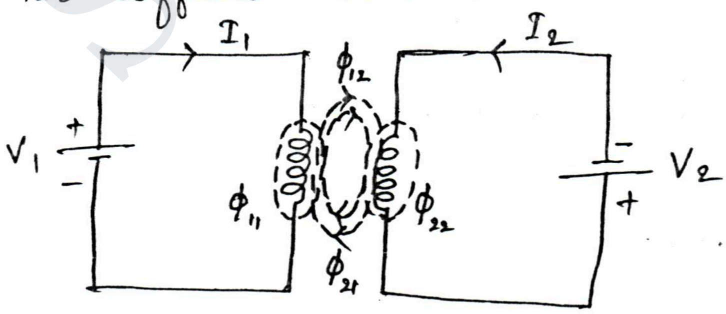
- "Inductance is defined as the ratio of total magnetic flux linkage to the current through the coil".
- denoted by L.
- Unit is Henry.
- If ϕ varies linearly with I

$$L = \frac{\text{Total flux Linkage}}{\text{Total Current}}$$

$$\boxed{L = \frac{N\phi}{I}} \text{ Henry.}$$

MUTUAL INDUCTANCE:

- Consider a magnetic circuit with two different coils carrying two different currents.



- Let coil 1 of N_1 turns with inductance L_1 carries current I_1. The magnetic flux produced by current I_1 flowing through coil 1 is ϕ_{11}
- Let coil 2 of N_2 turns with inductance L_2 carries current I_2. The magnetic flux produced by current I_2 flowing through coil 2 is ϕ_{22}.

 - The flux produced in coil 1 links with coil 2 and produces another flux ϕ_{12}.

- The flux produced in coil 2 links with coil 1 and produces a flux ϕ_{21}.

- The total flux linkage of the second coil due to the flux produced by current I_1 in first coil is $N_2 \phi_{12}$.

- The total flux linkage of the first coil due to the flux produced by current I_2 in second coil is $N_1 \phi_{21}$.

"Mutual Inductance between two coils is defined as the ratio of flux linkage of one coil to the current in other coil."

$$M_{12} = \frac{N_2 \phi_{12}}{I_1}$$

$$M_{21} = \frac{N_1 \phi_{21}}{I_2}$$

- If the magnetic circuit is linear $M_{12} = M_{21}$.

Inductance in Series:

(i) 

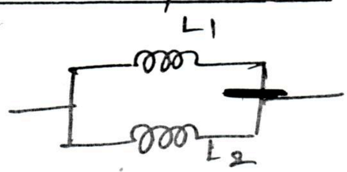
$$\boxed{L_{eq} = L_1 + L_2}$$

(ii) 

$$\boxed{L_{eq} = L_1 + L_2 + 2M}$$ If L_1 & L_2 are cumulatively or positively coupled.

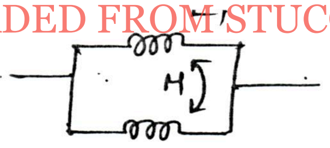
$$\boxed{L_{eq} = L_1 + L_2 - 2M}$$ If L_1 & L_2 are differentially or negatively coupled.

Inductance in Parallel:

(i) 

$$\boxed{L_{eq} = \frac{L_1 L_2}{L_1 + L_2}}$$

(ii)



L_2

$$\boxed{L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}}$$

If L_1 & L_2 are Cumulatively or Positively Coupled.

$$\boxed{L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}}$$

If L_1 & L_2 are differentialy or negatively Coupled.

PROBLEM.

Two Coils with negligible resistance and self inductances of 0.2 H and 0.1 H respectively are Connected in (i) Series (ii) Parallel. If their mutual inductance is 0.1 H, determine the effective inductance of their Combination in each case.

Given:

$L_1 = 0.2\ H$

$L_2 = 0.1\ H$

$M = 0.1\ H$

(i) Series Inductance

for positively Coupled

$$L_{eq} = L_1 + L_2 + 2M$$

$$= 0.2 + 0.1 + 2(0.1)$$

$$\boxed{L_{eq} = 0.5\ H}$$

for negatively Coupled

$$L_{eq} = L_1 + L_2 - 2M$$

$$= 0.2 + 0.1 - 2(0.1)$$

$$\boxed{L_{eq} = 0.1\ H}$$

(ii) Parallel Inductance

for Positively Coupled.

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

$$= \frac{(0.2)(0.1) - (0.1)^2}{0.2 + 0.1 - 2(0.1)}$$

$$\boxed{L_{eq} = 0.1\ H}$$

for negatively Coupled

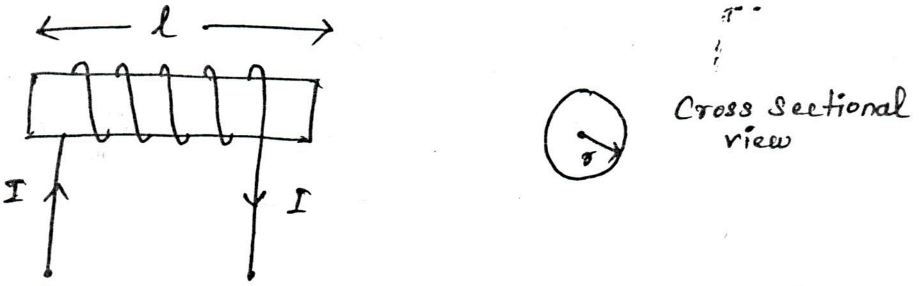
$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

$$= \frac{(0.2)(0.1) - (0.1)^2}{0.2 + 0.1 + 2(0.1)}$$

$$\boxed{L_{eq} = 0.02\ H}$$

INDUCTANCE OF A SOLENOID :

- Consider a solenoid of N - turns
- Let current flowing through the solenoid be I Ampere.
- Let the length of solenoid be l and the cross sectional area be A.



The magnetic field Intensity inside the Solenoid is

$$H = \frac{NI}{l} \ A/m \quad \text{—— (1)}$$

The total flux linkage is

$$N\phi = N(BA)$$

Sub $B = \mu H$

$$= N\mu H A \quad \text{—— (2)}$$

Sub (1) in (2)

$$\left.\begin{matrix}\text{Total flux}\\ \text{linkage}\end{matrix}\right\} = N\mu\left(\frac{NI}{l}\right)A$$

$$= \frac{\mu N^2 I A}{l} \quad \text{—— (3)}$$

$$\left.\begin{matrix}\text{The inductance}\\ \text{of a Solenoid}\end{matrix}\right\} L = \frac{\text{Total flux Linkage}}{\text{Total Current}}$$

$$= \frac{\left(\frac{\mu N^2 \cancel{I} A}{l}\right)}{\cancel{I}}$$

$$\boxed{L = \frac{\mu N^2 A}{l}} \ H$$

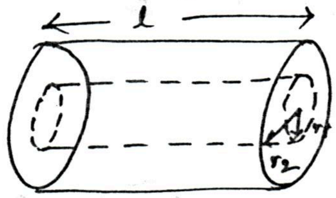
where

$A = \pi r^2 =$ Area of cross section of Solenoid

N - No. of turns

l - length of solenoid.

If two Solenoids are Connected in Coaxial manner; the inner Conductor with N_1 turns & radius of cross section r_1 and the outer Conductor with N_2 turns & radius of cross section r_2, (47)



Inductance of Inner Conductor } $L_{in} = \frac{\mu N_1^2 A_1}{l}$

Inductance of outer Conductor } $L_{out} = \frac{\mu N_2^2 A_2}{l}$

Mutual Inductance of a Solenoid } $M = \frac{\mu N_1 N_2 A_1}{l}$

PROBLEM

Calculate the Inductance of the Solenoid of 2500 turns wound uniformly over a length of 0.5 m on a cylindrical paper tube 4 cm in the dia. The medium is air.

Given: $N = 2500$

$l = 0.5\,m$

$r = 2\,cm = 2 \times 10^{-2}\,m$.

Solution:

Inductance of Solenoid } $L = \frac{\mu_0 N^2 A}{l}$ $(\because \mu_r = 1)$

where $A = \pi r^2$

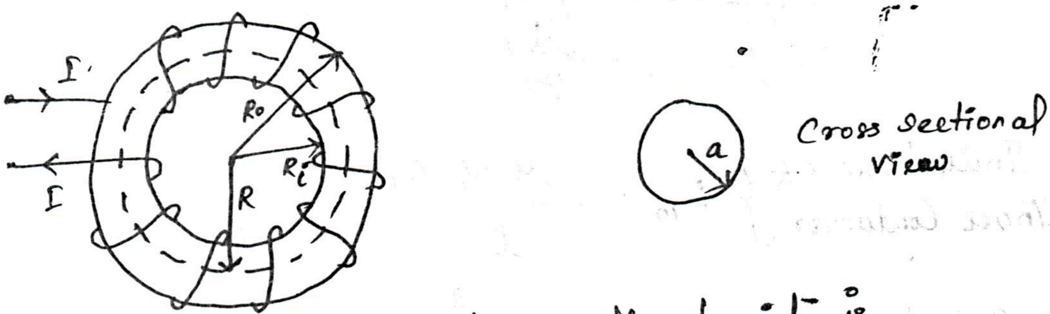
$$L = \frac{\mu_0 N^2 \pi r^2}{l}$$

$$= \frac{4\pi \times 10^{-7} \times (2500)^2 \times \pi \times (2 \times 10^{-2})^2}{0.5}$$

$$\boxed{L = 0.0197\,H}$$

INDUCTANCE OF A TOROID:

- Consider a toroidal ring with N - turns and carrying Current I.
- Let the radius of the toroid be R



The magnetic field intensity of the toroid is

$$H = \frac{NI}{2\pi R} \text{ A/m} \quad \text{——— (1)}$$

where $R = \frac{R_i + R_o}{2}$ = Mean radius

Total flux Linkage } $= N\phi$

$= N(BA)$

Sub $B = \mu H$

$= N\mu H A$ ——— (2)

Sub (1) in (2)

Total flux Linkage } $= N\mu\left(\frac{NI}{2\pi R}\right) A$

$$= \frac{\mu N^2 I A}{2\pi R} \quad \text{——— (3)}$$

Inductance of a toroid } $L = \frac{\text{Total flux linkage}}{\text{Total Current}}$

$$= \frac{\left(\frac{\mu N^2 \cancel{I} A}{2\pi R}\right)}{\cancel{I}}$$

$$\boxed{L = \frac{\mu N^2 A}{2\pi R}} \quad \text{——— (4)}$$

where

A – Area of cross section $= \pi a^2$

R – Mean radius of toroid

Sub $A = \pi a^2$ in (4)

$$L = \frac{\mu N^2 \times \cancel{\pi} a^2}{2\cancel{\pi} R}$$

$$\boxed{L = \frac{\mu N^2 a^2}{2R}} \quad \text{——— (5)}$$

PROBLEM.

A toroid of 1000 turns has a mean radius of 20 cm and radius for the winding of 2 cm. What is the inductance for

(i) with air core

(ii) with a iron core of $\mu_r = 800$.

Given:

$N = 1000$

$R = 20 \times 10^{-2}$ m

$a = 2 \times 10^{-2}$ m.

Solution:

(i) $(\mu_r = 1)$

$$L = \frac{\mu_0 N^2 a^2}{2R}$$

$$= \frac{4\pi \times 10^{-7} \times (1000)^2 \times (2\times10^{-2})^2}{2 \times 20 \times 10^{-2}}$$

$$\boxed{L = 1.257 \text{ mH}}$$

(ii) $\mu_r = 800$.

$$L = \frac{\mu_0 \mu_r N^2 a^2}{2R}$$

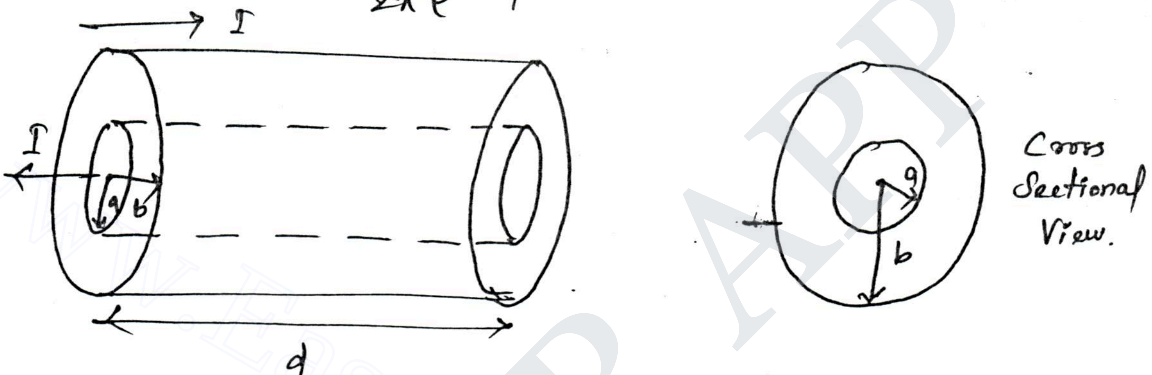
$$= \frac{4\pi \times 10^{-7} \times 800 \times (1000)^2 \times (2\times10^{-2})^2}{2 \times 20 \times 10^{-2}}$$

$$\boxed{L = 1.005\ H}$$

INDUCTANCE OF A COAXIAL CABLE :

- Consider a Coaxial cable with inner conductor radius 'a' and outer conductor radius 'b'.
- Let current through Coaxial Cable be I.
- In Coaxial Cable, the field Intensity at any point between inner & outer conductor is

$$\vec{H} = \frac{I}{2\pi\rho}\vec{a_\phi} \quad \text{——— (1)}$$



$$\vec{B} = \mu\vec{H} = \frac{\mu I}{2\pi\rho}\vec{a_\phi} \quad \text{——— (2)}$$

Here $\vec{a_\phi}$ is unit normal vector.

$$\therefore d\vec{S} = d\rho\, dz\, \vec{a_\phi} \quad \text{——— (3)}$$

$$\vec{B}\cdot d\vec{s} = \frac{\mu I}{2\pi\rho} d\rho\, dz$$

Total flux $\Big\}$ $$\phi = \int_S \vec{B}\cdot d\vec{S} = \int_0^d \int_a^b \frac{\mu I}{2\pi\rho} d\rho\, dz$$

$$= \frac{\mu I}{2\pi}\int_0^d [\ln\rho]_a^b\, dz$$

$$= \frac{\mu I}{2\pi}\ln\left(\frac{b}{a}\right)[z]_0^d$$

$$\phi = \frac{\mu I d}{2\pi}\ln\left(\frac{b}{a}\right)$$

Inductance of Coaxial Cable $\Big\}$ $L = \frac{\phi}{I}$

$$= \frac{\frac{\mu \cancel{I} d}{2\pi} \ln\left(\frac{b}{a}\right)}{\cancel{I}}$$

$$\boxed{L = \frac{\mu d}{2\pi} \ln\left(\frac{b}{a}\right)} \quad H$$

Inductance per length is

$$\boxed{\frac{L}{d} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right).} \quad H/m.$$

PROBLEM:

Calculate the inductance of a 10 m length of Coaxial Cable filled with a material for which $\mu_r = 80$ and radius of inner and outer Conductors are 1 mm and 4 mm respectively.

Given:

$d = 10 m$

$a = 1 \times 10^{-3} m$

$b = 4 \times 10^{-3} m$

$\mu_r = 80$.

Solution:

Inductance, $L = \frac{\mu d}{2\pi} \ln\left(\frac{b}{a}\right)$

$$= \frac{4\pi \times 10^{-7} \times 10}{2\pi} \ln\left(\frac{4 \times 10^{-3}}{1 \times 10^{-3}}\right)$$

$$= 2.218 \times 10^{-4} H.$$

$$\boxed{L = 221.8 \mu H}$$

Find the Inductance per unit length if radius of inner and outer Conductors are 1 mm and 8 mm respectively. Assume relative Permeability as unity.

Given:

$d = 1\ m$

$b = 8 \times 10^{-3}\ m$

$a = 1 \times 10^{-3}\ m$

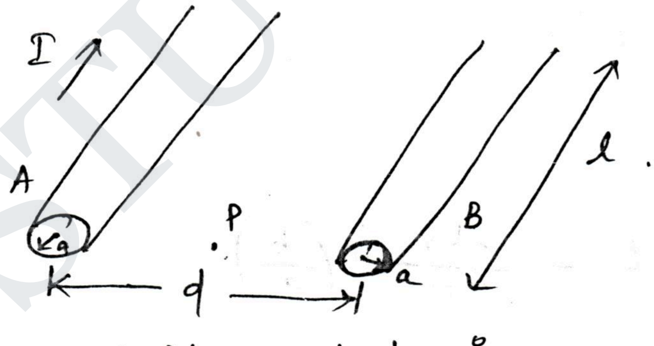
$$\left.\begin{array}{l}\text{Ind per}\\ \text{unit length}\end{array}\right\} = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$= \frac{4\pi \times 10^{-7}}{2\pi} \ln\left(\frac{8 \times 10^{-3}}{1 \times 10^{-3}}\right)$$

$$\boxed{L = 0.2197\ \mu H/m.}$$

INDUCTANCE FOR TRANSMISSION LINES:

- Consider a transmission line with two conductor having radius 'a' separated by 'd' and length is 'l'.
- Let the current through transmission line be I.



$\vec{H}$ of an infinite Conductor is

$$\vec{H} = \frac{I}{2\pi\rho} \vec{a_\phi} \quad \text{--- (1)}$$

$$\vec{B} = \frac{\mu I}{2\pi\rho} \vec{a_\phi} \quad \text{--- (2)}$$

$$\vec{ds} = d\rho\, dz\, \vec{a_\phi}$$

$$\vec{B} \cdot \vec{ds} = \frac{\mu I}{2\pi\rho} d\rho\, dz.$$

The total flux at Conductor A is

53

$$\phi_A = \int_s \frac{\mu I}{2\pi\rho} d\rho\, dz$$

$$= \frac{\mu I}{2\pi} \int_0^l \int_a^{d-a} \frac{d\rho}{\rho} dz$$

$$= \frac{\mu I}{2\pi} \int_0^l \left[\ln \rho\right]_a^{d-a} dz$$

$$= \frac{\mu I}{2\pi} \ln\left(\frac{d-a}{a}\right) \left[z\right]_0^l$$

$$\phi_A = \frac{\mu I l}{2\pi} \ln\left(\frac{d-a}{a}\right)$$

The inductance at Conductor A $\Big\}$ $L_A = \frac{\phi_A}{I} = \frac{\frac{\mu \cancel{I} l}{2\pi} \ln\left(\frac{d-a}{a}\right)}{\cancel{I}}$

$$L_A = \frac{\mu l}{2\pi} \ln\left(\frac{d-a}{a}\right)$$

The inductance per unit length is

$$L_A = \frac{\mu}{2\pi} \ln\left(\frac{d-a}{a}\right) \quad \text{——— (3)}$$

The inductance for Conductor B is same as the inductance of Conductor A.

$$L_B = \frac{\mu}{2\pi} \ln\left(\frac{d-a}{a}\right) \quad \text{———→ (4)}$$

Total Inductance $\Big\}$ $L = L_A + L_B$

$$= \frac{\mu}{\cancel{2}\pi}\left[\cancel{2} \ln\left(\frac{d-a}{a}\right)\right]$$

$$\boxed{L = \frac{\mu}{\pi} \ln\left(\frac{d-a}{a}\right)} \quad H.$$

Behaviour of Magnetic Materials :

Basically the magnetic materials are classified on the basis of presence of magnetic dipole moment in materials.

A charged particle with angular momentum always contribute to permanent magnetic dipole momentum.

a) Orbital magnetic dipole moment
b) Electron spin magnetic moment
c) Nuclear spin magnetic moment.

On the basis of magnetic behaviour the magnetic materials are classified as

1. Diamagnetic
2. Paramagnetic
3. Ferromagnetic
4. Antiferromagnetic
5. Ferrimagnetic

Diamagnetic :

→ Magnetic materials in which the orbital magnetic moment and electron spin moment cancel each other, making net permanent magnetic momentum of each atom is zero and are called diamagnetic materials.

→ Materials which lack permanent dipoles are called ~~diag~~ diamagnetic materials.

→ when an external magnetic field is applied, the motion of an electron in their orbit changes, which results an induced magnetic moment in a direction opposite to the direction of applied field.

Properties:

i) $\chi_m < 0$, $\mu_r \leq 1$

ii) permanent Dipoles are absent.

iii) Temperature Independent.

iv) Linear magnetic materials

v) when placed inside a magnetic field, magnetic lines of forces are repeled

Ex: Silicon, Diamond, Lead, Copper.

2. Paramagnetic Materials:

* Each Electron in an orbit has an orbital magnetic moment and spin magnetic moment
* They do not cancel each other.
* In the paramagnetic materials, atoms are oriented randomly.

Properties:

i) $\chi_m > 0$; $\mu_r \geq 1$

ii) Linear Magnetic materials.

iii) Permanent magnetic dipoles are present.

iv) Temperature dependent.

v) In the absence of external applied field, the dipoles are randomly oriented. Hence net magnetization is zero.

vi) When an external field is applied, then each atomic dipole moment experiences a torque.

vii) Spin Alignments are random ↗ ↘ ↘ ←

Eg: Potassium, tungston, ~~Oxygen~~.

3. Ferromagnetic Materials:

The materials in which atoms have large dipole moment due to electron spin magnetic moments are called Ferromagnetic materials.

Properties:

i) $\chi_m >> 0$, $\mu_r >> 1$

ii) Spin Alignment is parallel ↑↑↑↑

iii) It posses large permanent dipole moment.

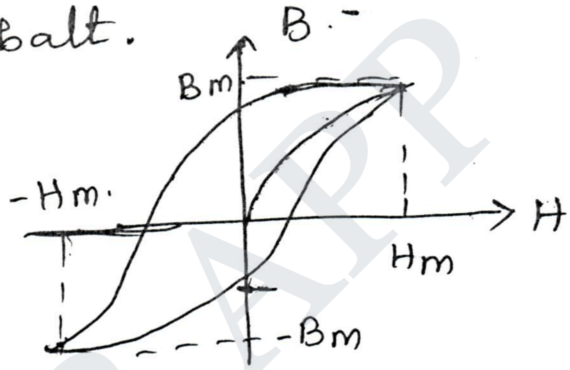
iv) The region in which large no. of magnetic moments lined in parallel are called domains.

v) When external field is applied, the domain increase their size increasing internal field to a high value.

vi) when the external field is removed, the original random alignment of dipole moments is not achieved.

Some of the moments remain in a small region which results in residual field (or) remanant field. This effect is called hysteresis.

Eg: Iron, Nickel, cobalt.



Hysteresis Loop.

4. Antiferromagnetic Materials:

The Materials in which the dipole moments of adjacent atoms line up in antiparallel fashion are called antiferromagnetic materials.

↑ ↓ ↑ ↓ (atomic spin Alignment).

Therefore net magnetic moment is zero.

Properties:

i) Temperature dependent.
ii) spin Alignment is antiparallel.
iii) Eg: chloride, sulphide, oxide.

5. Ferrimagnetic materials:

The materials in which the magnetic dipole moments are lined up in antiparallel but with different magnitude are called Ferrimagnetic materials.

↑↓↑↓↑↓ — Spin Alignment.

Properties:

i) Ferrites are special case of Ferrimagnets.

ii) It posses net magnetic moment.

iii) Eg: Nickel ferrite, nickel-zinc-Ferrite.

MAGNETIC FORCES AND TORQUES:

MAGNETIC FORCES:

~~when~~ A magnetic force ($\vec{F_m}$) exerted on a charge Q, moving with a velocity $\vec{v}$ in a steady magnetic field $\vec{B}$ is given by.

$$\vec{F_m} = Q[\vec{V} \times \vec{B}] \cdot (N).$$

Force on a Differential current Element:

Hall Effect:

Consider a conductor in which electrons are in motion. when a magnetic field is applied, a force exerted on the electrons and a small displacement takes place between positive and negative charges.

Thus the separation of charges is observed which indicates that small potential difference exists across conductor in a direction perpendicular to magnetic Field and velocity of charges. This small voltage across conductor is called Hall voltage

FORCE ON A DIFFERENTIAL CURRENT ELEMENT:

The differential force on moving differential charge dQ is

$$d\vec{F} = dQ(\vec{v} \times \vec{B}) \quad \text{--- (1)}$$

The Current density $\vec{J}$ can be expressed interms of velocity $(\vec{v})$ and volume charge density (ρ_v) as

$$\vec{J} = \rho_v \vec{v} \quad \text{--- (2)}$$

The differential element of charge dQ can be expressed interms of volume charge density is.

$$dQ = \rho_v \, dV \quad \text{--- (3)}$$

Sub eqn (3) in (1)

$$d\vec{F} = \rho_v \, dv (\vec{v} \times \vec{B})$$

from eqn (2)

$$\boxed{d\vec{F} = (\vec{J} \times \vec{B}) \, dv} \quad \text{--- (4)}$$

$$\boxed{\vec{F} = \int_v (\vec{J} \times \vec{B}) \, dv} \quad \text{--- (5)}$$

← Force on a differential Volume.

WKT $\vec{J} dv = \vec{K} ds$

(4) ⇒ $$\boxed{d\vec{F} = (\vec{K} \times \vec{B}) \, ds} \quad \text{--- (6)}$$

$$\boxed{\vec{F} = \int_s (\vec{K} \times \vec{B}) \, ds} \quad \text{--- (7)}$$

← Force on a differential Surface.

WKT $\vec{J} dv = I d\vec{l}$

(4) ⇒ $$\boxed{d\vec{F} = I \, d\vec{l} \times \vec{B}} \quad \text{--- (8)}$$

$$\boxed{\vec{F} = \int I \, d\vec{l} \times \vec{B}} \quad \text{--- (9)}$$

← Force on a differential length

If the Conductor has uniform field $\vec{B}$ and straight, then the force will be

$$\boxed{\vec{F} = I(\vec{L} \times \vec{B})} \quad \text{--- (10)}$$

$$\vec{F} = ILB \sin\theta \, \hat{n}$$

$$\boxed{F = |\vec{F}| = ILB \sin\theta} \quad \text{--- (11)}$$

PROBLEM

(2) A Conductor of 6m long, lies along z-direction with a Current of 2A in $\vec{a_z}$ direction. Find the force experienced by the conductor if $\vec{B} = 0.08\,\vec{a_x}$ Tesla.

Given: $I = 2A$

$\vec{dl} = 6\,\vec{a_z}$

$\vec{B} = 0.08\,\vec{a_x}$

Solution

$$\vec{F} = I\,\vec{dl} \times \vec{B}$$

$$= 2(6\,\vec{a_z} \times 0.08\,\vec{a_x})$$

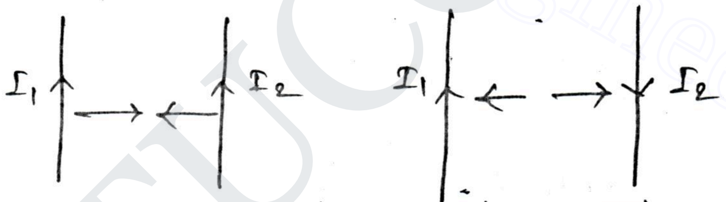
$$= 2 \times 0.48\,(\vec{a_z} \times \vec{a_x})$$

$$\boxed{\vec{F} = 0.96\,\vec{a_y}\ \text{N}}$$

FORCE BETWEEN DIFFERENTIAL CURRENT ELEMENTS:

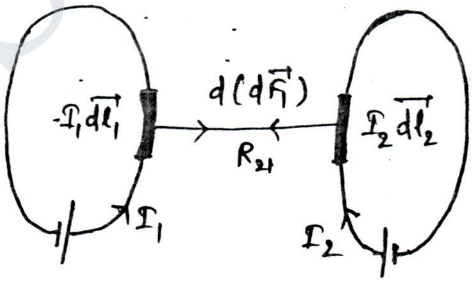
— If the Current through the parallel Conductors are in same direction, the force between the two conductors is attractive force.

— If the Current through the parallel Conductors are in opposite direction, the force between the two conductors is repulsive force.



— Consider two Current elements $I_1\vec{dl_1}$ and $I_2\vec{dl_2}$.

— The Currents I_1 and I_2 are in same direction.



— The force exerted $d(d\vec{F_1})$ on element $I_1\vec{dl_1}$ due to the magnetic field $d\vec{B_2}$ produced by other element $I_2\vec{dl_2}$ is the force of attraction.

$$d(d\vec{F_1}) = I_1\,\vec{dl_1} \times d\vec{B_2} \quad \text{——— (1)}$$

By Biot Savart law

$$d\vec{H_2} = \frac{I_2 \vec{dl_2} \times \vec{a_{R_{21}}}}{4\pi R_{21}^2}$$

$$d\vec{B_2} = \mu \, d\vec{H_2}$$

$$d\vec{B_2} = \mu \left[\frac{I_2 \vec{dl_2} \times \vec{a_{R_{21}}}}{4\pi R_{21}^2} \right] \quad \text{——— (2)}$$

Sub (2) in (1)

$$d(d\vec{F_1}) = I_1 \vec{dl_1} \times \mu \left[\frac{I_2 \vec{dl_2} \times \vec{a_{R_{21}}}}{4\pi R_{21}^2} \right]$$

$$= \mu \left[\frac{I_1 \vec{dl_1} \times (I_2 \vec{dl_2} \times \vec{a_{R_{21}}})}{4\pi R_{21}^2} \right]$$

By Integrating twice

$$\vec{F_1} = \frac{\mu I_1 I_2}{4\pi} \oint_{l_1} \oint_{l_2} \frac{\vec{dl_1} \times (\vec{dl_2} \times \vec{a_{R_{21}}})}{R_{21}^2} \quad \text{——— (3)}$$

IIIly

$$\vec{F_2} = \frac{\mu I_2 I_1}{4\pi} \oint_{l_2} \oint_{l_1} \frac{\vec{dl_2} \times (\vec{dl_1} \times \vec{a_{R_{12}}})}{R_{12}^2} \quad \text{——— (4)}$$

$$\therefore \boxed{\vec{F_2} = -\vec{F_1}}$$

∴ For the current carrying conductors of length l each, with distance of separation d, the force exerted is given by

$$\boxed{F = \frac{\mu I_1 I_2 l}{2\pi d}} \quad \text{——— (5)}$$

PROBLEM:

(3) A current element $I_1 \vec{dl_1} = 10^{-5} \vec{a_z}$ Am is located at $P_1(1,0,0)$ while a second element $I_2 \vec{dl_2} = 10^{-5}(0.6\vec{a_x} - 2\vec{a_y} + 3\vec{a_z})$ Am is at $P_2(-1,0,0)$ both in free space. Find the vector force exerted on $I_2 \vec{dl_2}$ by $I_1 \vec{dl_1}$.

Given:

$$I_1 \vec{dl_1'} = 10^{-5}\,\vec{a_z}$$

$$\vec{R_1'} = \vec{a_x}$$

$$I_2 \vec{dl_2'} = 10^{-5}\left(0.6\vec{a_x} - 2\vec{a_y} + 3\vec{a_z}\right)$$

$$\vec{R_2'} = -\vec{a_x}$$

Solution:

$$\vec{R_{12}} = \vec{R_2'} - \vec{R_1'} = -2\vec{a_x}$$

$$R_{12} = |\vec{R_{12}}| = 2$$

$$\vec{a_{R_{12}}} = \frac{\vec{R_{12}}}{R_{12}} = \frac{-2\vec{a_x}}{2} = -\vec{a_x}$$

$$\vec{dH_1'} = \frac{I_1 \vec{dl_1'} \times \vec{a_{R_{12}}}}{4\pi R_{12}^2} = \frac{(10^{-5}\vec{a_z}) \times (-\vec{a_x})}{4\pi (2)^2}$$

$$\vec{dH_1'} = \frac{-10^{-5}}{16\pi}\vec{a_y} \text{ A/m}, \quad \vec{dB_1'} = \mu_0 \vec{dH_1'} = -0.25 \times 10^{-12}\,\vec{a_y} \text{ Wb/m}^2$$

The force exerted on $I_2 \vec{dl_2'}$ due to $I_1 \vec{dl_1'}$ is

$$\text{Force} = I_2 \vec{dl_2'} \times \vec{dB_1'}$$

$$= 10^{-5}\left(0.6\vec{a_x} - 2\vec{a_y} + 3\vec{a_z}\right) \times \left(-0.025 \times 10^{-12}\,\vec{a_y}\right)$$

$$\boxed{\vec{F} = \left(7.5\,\vec{a_x} - 1.5\,\vec{a_z}\right) \times 10^{-18} \text{ N}}$$

(4) Find the force between two long parallel wires A and B separated by 5 cm in air carrying current of $I_1 = 8$ A, $I_2 = 6$ A. Length of each conductor is 200 m.

Given: $I_1 = 8$ A, $I_2 = 6$ A, $d = 5 \times 10^{-2}$ m, $l = 200$ m

$$F = \frac{\mu_0 I_1 I_2 l}{2\pi d}$$

$$= \frac{\cancel{4}^{2}\cancel{\pi} \times 10^{-7} \times 8 \times 6 \times \cancel{200}^{40}}{\cancel{2\pi} \times \cancel{5} \times 10^{-2}}$$

$$\boxed{F = 14400\ \mu\text{N}}$$

FORCE AND TORQUE ON A CLOSED CIRCUIT:

The force on a Current element $I\vec{dl}$ is closed circuit is

$$\vec{F} = \oint I\vec{dl} \times \vec{B} = -\oint \vec{B} \times I\vec{dl}$$

$$= -I\oint \vec{B} \times \vec{dl}$$

for uniform magnetic flux density

$$\vec{F} = -IB\oint \vec{dl}$$

The closed line integral is a field is zero. (i,e) $\oint \vec{dl} = 0$

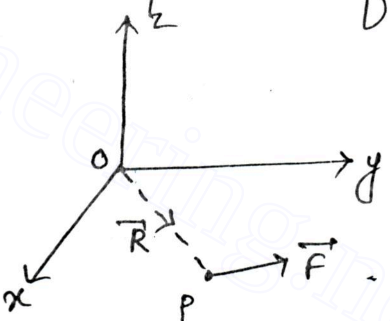
$$\boxed{\therefore \vec{F} = 0}$$

(i,e) The force on a closed filamentary circuit in a uniform magnetic field is zero.

- If the field is not uniform, $\vec{F}$ need not be zero.
- For a closed circuit force is zero and Torque is not equal to zero.

Torque or Moment of force:

Torque or Moment of force is the vector product of moment arm $(\vec{R})$ and Force $(\vec{F})$.



$$\boxed{\vec{T} = \vec{R} \times \vec{F}}$$

$\vec{T}$ is normal to $\vec{R}$ and $\vec{F}$.

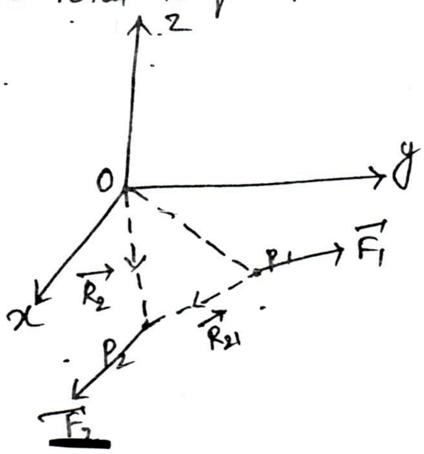
- Consider two forces $\vec{F_1}$ and $\vec{F_2}$ at point P_1 and P_2. The moment arm with respect to origin is $\vec{R_1}$ and $\vec{R_2}$. The Total torque $\vec{T}$ with respect to origin is

$$\vec{T} = (\vec{R_1} \times \vec{F_1}) + (\vec{R_2} \times \vec{F_2})$$

The total force on a closed circuit is zero.

$$\vec{F_1} + \vec{F_2} = 0$$

$$\vec{F_2} = -\vec{F_1}$$



$$\therefore \vec{T} = \vec{R_1} \times \vec{F_1} - \vec{R_2} \times \vec{F_1}$$

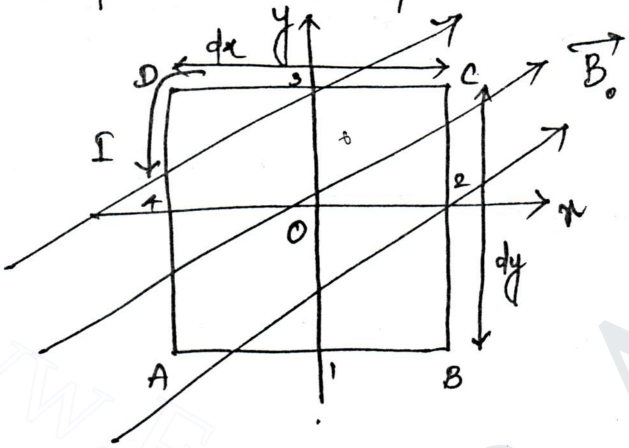
$$= (\vec{R_1} - \vec{R_2}) \times \vec{F_1}$$

$$\boxed{\vec{T} = \vec{R_{21}} \times \vec{F_1}}$$

$\because \vec{R_{21}} = \vec{R_1} - \vec{R_2}$.

TORQUE ON A DIFFERENTIAL CURRENT LOOP IN A MAGNETIC FIELD $\vec{B_0}$:

- Consider a differential current loop with uniform magnetic field $\vec{B_0}$ is placed in xy plane.
- dx and dy be the lengths of the sides of the loop.
- Origin of the coordinate system is centre of the loop.
- The value of the magnetic field at the centre of the loop carrying current I in anticlock direction be $\vec{B_0}$.
- The total force on the loop is zero.



The force exerted on side AB $\Big\}$ $d\vec{F_1} = I\,dx\,\vec{a_x} \times \vec{B_0}$

$$= I\,dx\,\vec{a_x} \times (B_{0x}\vec{a_x} + B_{0y}\vec{a_y} + B_{0z}\vec{a_z})$$

$$d\vec{F_1} = I\,dx\,(B_{0y}\vec{a_z} - B_{0z}\vec{a_y}) \quad \text{——— (1)}$$

For the line AB moment arm is $\vec{R_1}$

$$\vec{R_1} = \frac{1}{2}dy(-\vec{a_y}) = \frac{-1}{2}dy\,\vec{a_y} \quad \text{——— (2)}$$

$\therefore$ The torque on AB is

$$d\vec{T_1} = \vec{R_1} \times d\vec{F_1}$$

$$= \left[\frac{-1}{2}dy\,\vec{a_y}\right] \times \left[I\,dx\,(B_{0y}\vec{a_z} - B_{0z}\vec{a_y})\right]$$

$$d\vec{T_1} = -\frac{1}{2}B_{0y}\,I\,dx\,dy\,\vec{a_x} \quad \text{——— (A)}$$

The force exerted on side BC $\Big\}$ $d\vec{F_2} = I\,dy\,\vec{a_y} \times \vec{B_0}$

$$= I\,dy\,\vec{a_y} \times (B_{0x}\vec{a_x} + B_{0y}\vec{a_y} + B_{0z}\vec{a_z})$$

$$d\vec{T_1} = I\,dy\,(-B_{0x}\vec{a_z} + B_{0z}\vec{a_x}) \quad \text{——— (3)}$$

For the line BC moment arm is $\vec{R_2}$.

PROBLEM:

A loop with magnetic dipole moment $8 \times 10^{-3} \vec{a}_z$ Am^2, lies in a uniform magnetic field $\vec{B} = 0.2\vec{a}_x + 0.4\vec{a}_z$ Wb/m^2. Find the torque.

Given:

$$\vec{m} = 8 \times 10^{-3} \vec{a}_z \quad Am^2$$

$$\vec{B} = 0.2\vec{a}_x + 0.4\vec{a}_z \quad Wb/m^2$$

Solution:

$$\vec{T} = \vec{m} \times \vec{B}$$

$$= (8 \times 10^{-3} \vec{a}_z) \times (0.2\vec{a}_x + 0.4\vec{a}_z)$$

$$\boxed{\vec{T} = 1.6 \times 10^{-3} \vec{a}_y} \quad N.m$$

MAGNETIC FIELD INTENSITY AND IDEA OF RELATIVE PERMEABILITY

MAGNETIZATION AND PERMEABILITY:

– The current produced by the bound charges (orbital electrons, electron spin, nuclear spin) is called bound current or Amperian current. Represented as I_b.

– The bound charges are charges which are bound to nucleus.

– The field produced due to the movement of bound charges is called magnetization represented as $\vec{M}$.

– Let I_b flows through a closed path. Assume a closed path encloses a differential area $d\vec{s}$. Therefore the magnetic dipole moment is,

$$\vec{m} = I_b \, d\vec{s} \quad \text{——— (1)}$$

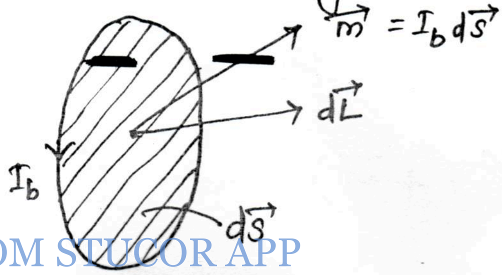
– For a differential volume, the total magnetic dipole moment is the summation of individual magnetic dipole moment of each magnetic dipole.

$$\vec{m}_{total} = \sum_{a=1}^{n\Delta V} \vec{m}_a \quad \text{——— (2)}$$

"Magnetization is defined as the magnetic dipole moment per unit volume"

$$\vec{M} = \lim_{\Delta V \to 0} \frac{\sum_{a=1}^{n\Delta V} \vec{m}_a}{\Delta V} \quad \text{——— (3)}$$

Alignment of magnetic dipole along a closed path:



- The differential volume can be represented as $d\vec{S}.d\vec{L}$.
- In this volume there are 'n' $d\vec{S}.d\vec{L}$ magnetic moments.
- With the external magnetic field, random orientation changes to partial alignment of the magnetic moments.
- To acheive this the bound current increased by I_b for each magnetic dipole.

$$\therefore dI_b = n I_b \, d\vec{S}.d\vec{L}$$

$$dI_b = \vec{M}.d\vec{L}$$

$$I_b = \oint \vec{M}.d\vec{L} \qquad \text{——— (4)}$$

- Here the total current is the sum of Bound current and free current.

$$I_T = I_b + I = \oint \vec{H}.d\vec{L} \qquad \text{——— (5)}$$

where I_b – Bound current

I – free electron current.

$$(5) \Rightarrow \quad I_T = \oint \frac{\vec{B}}{\mu_0}.d\vec{L} \qquad \text{——— (6)}$$

$\because \vec{B} = \mu_0 \vec{H}$

$\vec{H} = \frac{\vec{B}}{\mu_0}$

from equation (5)

$$I = I_T - I_b$$

Sub eqn (4) and (6)

$$I = \oint \frac{\vec{B}}{\mu_0}.d\vec{L} - \oint \vec{M}.d\vec{L}$$

$$I = \oint \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right).d\vec{L} \qquad \text{——— (7)}$$

WKT

$$I = \oint \vec{H}.d\vec{L} \qquad \text{——— (8)}$$

By comparing equation (7) & (8)

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

$$\vec{H} + \vec{M} = \frac{\vec{B}}{\mu_0}$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \qquad \text{——— (9)}$$

for linear, isotrophic magnetic materials,

$$\vec{M} = \chi_m \vec{H} \quad \text{--- (10)}$$

where χ_m – magnetic Susceptibility.

By substituting (10) in (9)

$$\vec{B} = \mu_0 [\vec{H} + \chi_m \vec{H}]$$

$$\vec{B} = \mu_0 (1+\chi_m) \vec{H} \quad \text{--- (11)}$$

The relation between $\vec{B}$ and $\vec{H}$ is

$$\vec{B} = \mu \vec{H} \quad \text{--- (12)}$$

$$\vec{B} = \mu_0 \mu_r \vec{H}$$

By comparing (11) and (12)

$$\boxed{\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}} \quad \text{--- (13)}$$

PROBLEM.

Find the Permeability of the material whose magnetic Susceptibility is 49.

Given:

$$\chi_m = 49$$

Solution:

$$\mu_r = 1 + \chi_m$$

$$= 1 + 49$$

$$\boxed{\chi_m = 50}$$

Permeability, $\mu = \mu_0 \mu_r$

$$= 4\pi \times 10^{-7} \times 50$$

$$\boxed{\mu = 6.28 \times 10^{-5}}$$

UNIT-IV

TIME-VARYING FIELDS AND MAXWELL'S EQUATION

5.1Introduction

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

$$\nabla \times \vec{E} = 0 \quad (5.1a)$$

$$\nabla \cdot \vec{D} = \rho_v \quad (5.1b)$$

For a linear and isotropic medium,

$$\vec{D} = \varepsilon \vec{E} \quad (5.1c)$$

Similarly for the magnetostatic case

$$\nabla . \vec{B} = 0 \quad (5.2a)$$

$$\nabla \times \vec{H} = \vec{J} \quad (5.2b)$$

$$\vec{B} = \mu \vec{H} \quad (5.2c)$$

It can be seen that for static case, the electric field vectors $\vec{E}$ and $\vec{D}$ and magnetic field vectors $\vec{B}$ and $\vec{H}$ form separate pairs.

In this chapter we will consider the time varying scenario. In the time varying case we will observe that a changing magnetic field will produce a changing electric field and vice versa.

We begin our discussion with Faraday's Law of electromagnetic induction and then present the Maxwell's equations which form the foundation for the electromagnetic theory.

4.1 Faraday's Law of electromagnetic Induction

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law. Mathematically, the induced emf can be written as

$$Emf = -\frac{d\phi}{dt} \quad Volts \qquad (5.3)$$

where ϕ is the flux linkage over the closed path.

A non zero $\frac{d\phi}{dt}$ may result due to any of the following:

(a) time changing flux linkage a stationary closed path.

(b) relative motion between a steady flux a closed path.

(c) a combination of the above two cases.

The negative sign in equation (5.3) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

$$Emf = -N\frac{d\phi}{dt} \quad Volts \qquad (5.4)$$

By defining the total flux linkage as

$$\lambda = N\phi \qquad (5.5)$$

The emf can be written as

$$Emf = -\frac{d\lambda}{dt} \qquad (5.6)$$

Continuing with equation (5.3), over a closed contour '*C*' we can write

$$Emf = \oint_C \vec{E}.d\vec{l} \qquad (5.7)$$

where $\vec{E}$ is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour '*C*' is given by

$$\phi = \int_S \vec{B}.d\vec{s} \qquad (5.8)$$

Where *S* is the surface for which '*C*' is the contour.

From (5.7) and using (5.8) in (5.3) we can write

$$\oint_C \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\oint_S \vec{B}.d\vec{s} \qquad (5.9)$$

By applying stokes theorem

$$\int_S \nabla\times\vec{E}.d\vec{s} = -\int_S \frac{\partial\vec{B}}{\partial t}.d\vec{s} \qquad (5.10)$$

Therefore, we can write

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (5.11)$$

which is the Faraday's law in the point form

We have said that non-zero $d\phi/dt$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a **transformer emf .**

Example: Ideal transformer

As shown in figure 5.1, a transformer consists of two or more numbers of coils coupled magnetically through a common core. Let us consider an ideal transformer whose winding has zero resistance, the core having infinite permittivity and magnetic losses are zero.

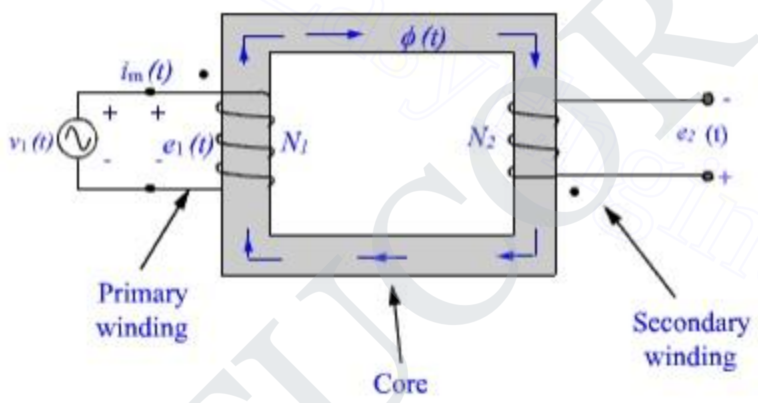


Fig 5.1: Transformer with secondary open

These assumptions ensure that the magnetization current under no load condition is vanishingly small and can be ignored. Further, all time varying flux produced by the primary winding will follow the magnetic path inside the core and link to the secondary coil without any leakage. If N_1 and N_2 are the number of turns in the primary and the secondary windings respectively, the induced emfs are

$$e_1 = N_1 \frac{d\phi}{dt} \qquad (5.12a)$$

$$e_2 = N_2 \frac{d\phi}{dt} \qquad (5.12b)$$

(The polarities are marked, hence negative sign is omitted. The induced emf is +ve at the dotted end of the winding.)

$$\therefore \frac{e_1}{e_2} = \frac{N_1}{N_2} \qquad (5.13)$$

i.e., the ratio of the induced emfs in primary and secondary is equal to the ratio of their turns. Under ideal condition, the induced emf in either winding is equal to their voltage rating.

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} = a \qquad (5.14)$$

where '*a*' is the transformation ratio. When the secondary winding is connected to a load, the current flows in the secondary, which produces a flux opposing the original flux. The net flux in the core decreases and induced emf will tend to decrease from the no load value. This causes the primary current to increase to nullify the decrease in the flux and induced emf. The current continues to increase till the flux in the core and the induced emfs are restored to the no load values. Thus the source supplies power to the primary winding and the secondary winding delivers the power to the load. Equating the powers

$$i_1 v_1 = i_2 v_2 \qquad (5.15)$$

$$\frac{i_2}{i_1} = \frac{v_1}{v_2} = \frac{e_1}{e_2} = \frac{N_1}{N_2} \qquad (5.16)$$

Further,

$$i_2 N_2 - i_1 N_1 = 0 \qquad (5.17)$$

i.e., the net magnetomotive force (mmf) needed to excite the transformer is zero under ideal condition.

Motional EMF:

Let us consider a conductor moving in a steady magnetic field as shown in the fig 5.2.

Fig 5.2

If a charge Q moves in a magnetic field $\vec{B}$, it experiences a force

$$\vec{F} = Q\vec{v} \times \vec{B} \qquad (5.18)$$

This force will cause the electrons in the conductor to drift towards one end and leave the other end positively charged, thus creating a field and charge separation continuous until electric and magnetic forces balance and an equilibrium is reached very quickly, the net force on the moving conductor is zero.

$\frac{\vec{F}}{Q} = \vec{v} \times \vec{B}$ can be interpreted as an induced electric field which is called the motional electric field

$$\vec{E}_m = \vec{v} \times \vec{B} \qquad (5.19)$$

If the moving conductor is a part of the closed circuit C, the generated emf around the circuit is $\oint_c \vec{v} \times \vec{B}\, d\vec{l}$. This emf is called the **motional emf.**

5.2 Maxwell's Displacement Current

Common thing that passes through the surface and between the capacitor plates is an electric field. This field is perpendicular to the surface, has the same magnitude over the area of the capacitor plats and vanishes outside it.

Hence, the electric flux through the surface is Q/ε_0 (using Gauss's law). Further, since the charge on the capacitor plates changes with time, for consistency we can calculate the current as follows:

$$i = \varepsilon_0 \,(dQ/dt)$$

This is the missing term in Ampere's circuital law. In simple words, when we add a term which is ε_0 times the rate of change of electric flux to the total current carried by the conductors, through the same surface, then the total has the same value of current 'i' for all surfaces. Therefore, no contradiction is observed if we use the Generalized Ampere's Law.

Hence, the magnitude of B at a point P outside the plates is the same at a point just inside. Now, the current carried by conductors due to the flow of charge is called 'Conduction current'. The new term added is the current that flows due to the changing electric field and is called 'Displacement current' or Maxwell's Displacement current'.

Hence, the magnitude of B at a point P outside the plates is the same at a point just inside. Now, the current carried by conductors due to the flow of charge is called 'Conduction current'. The new term added is the current that flows due to the changing electric field and is called 'Displacement current' or Maxwell's Displacement current'.

Displacement Current Explained

By now we understand that there are two sources of a magnetic field:

1. Conduction electric current due to the flow of charges
2. Displacement current due to the rate of change of the electric field

Hence, the total current (i) is calculated as follows: (where i_c – conduction current and i_d – displacement current)

$$i = i_c + i_d$$

$$= i_c + \varepsilon_0(dQ/dt)$$

This means that –

- Outside the capacitor plates: $i_c=i$ and $i_d=0$
- Inside the capacitor plates: $i_c=0$ and $i_d=i$

So, the generalized Ampere's law states:

The total current passing through any surface of which the closed loop is the perimeter is the sum of the conduction current and the displacement current.

This is also known as – **Ampere-Maxwell Law**. It is important to remember that the displacement and conduction currents have the same physical effects. Here are some points to remember:

- In cases where the electric field does not change with time, like steady electric fields in a conducting wire, the displacement current may be zero.
- In cases like the one explained above, both currents are present in different regions of the space.
- Since a perfectly conducting or insulating medium does not exist, in most cases both the currents can be present in the same region.
- In cases where there is no conduction current but a time-varying electric field, only displacement current is present. In such a scenario we have a magnetic field even when there is no conduction current source nearby.

5.3 Faraday's Law of Induction and Ampere-Maxwell Law

According to Faraday's law of induction, there is an induced emf which is equal to the rate of change of magnetic flux. Since emf between two points is the work done per unit charge to take it from one point to the other, its existence simply implies the existence of an electric field. Rephrasing Faraday's law:

A magnetic field that changes with time gives rise to an electric field.

Hence, an electric field changing with time gives rise to a magnetic field. This is a consequence of the displacement current being the source of the magnetic field. Hence, it is fair to say that time-dependent magnetic and electric fields give rise to each other

5.4 Maxwell's Equation

Equation (5.1) and (5.2) gives the relationship among the field quantities in the static field. For time varying case, the relationship among the field vectors written as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (5.20a)$$

$$\nabla \times \vec{H} = \vec{J} \qquad (5.20b)$$

$$\nabla . \vec{D} = \rho \qquad (5.20c)$$

$$\nabla . \vec{B} = 0 \qquad (5.20d)$$

In addition, from the principle of conservation of charges we get the equation of continuity

$$\nabla . \vec{J} = -\frac{\partial \rho}{\partial t} \qquad (5.21)$$

The equation 5.20 (a) - (d) must be consistent with equation (5.21).

We observe that

$$\nabla . \nabla \times \vec{H} = 0 = \nabla . \vec{J} \qquad (5.22)$$

Since $\nabla . \nabla \times \vec{A}$ is zero for any vector $\vec{A}$.

Thus $\nabla \times \vec{H} = \vec{J}$ applies only for the static case i.e., for the scenario when $\frac{\partial \rho}{\partial t} = 0$.

A classic example for this is given below . Suppose we are in the process of charging up a capacitor as shown in fig 5.3.

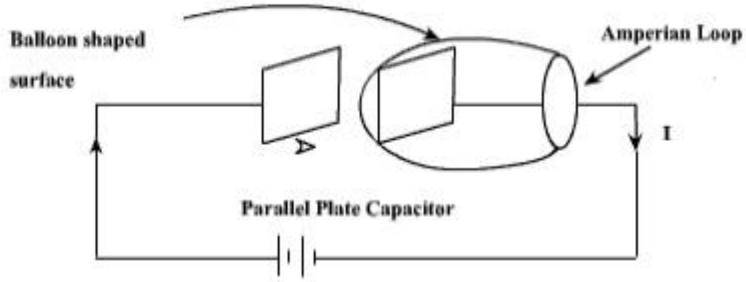


Fig 5.3

Let us apply the Ampere's Law for the Amperian loop shown in fig 5.3. $I_{enc} = I$ is the total current passing through the loop. But if we draw a baloon shaped surface as in fig 5.3, no current passes through this surface and hence $I_{enc} = 0$. But for non steady currents such as this one, the concept of current enclosed by a loop is ill-defined since it depends on what surface you use. In fact Ampere's Law should also hold true for time varying case as well, then comes the idea of displacement current which will be introduced in the next few slides.

We can write for time varying case,

$$\nabla.\left(\nabla\times\vec{H}\right) = 0 = \nabla.\vec{J} + \frac{\partial\rho}{\partial t}$$

$$= \nabla.\vec{J} + \frac{\partial}{\partial t}\nabla.\vec{D}$$

$$= \nabla.\left(\vec{J} + \frac{\partial\vec{D}}{\partial t}\right) \qquad (5.23)$$

$$\therefore \nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad (5.24)$$

The equation (5.24) is valid for static as well as for time varying case.

Equation (5.24) indicates that a time varying electric field will give rise to a magnetic field even in the absence of $\vec{J}$. The term $\frac{\partial\vec{D}}{\partial t}$ has a dimension of current densities (A/m^2) and is called the displacement current density.

Introduction of $\frac{\partial \vec{D}}{\partial t}$ in $\nabla \times \vec{H}$ equation is one of the major contributions of Jame's Clerk Maxwell. The modified set of equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (5.25a)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad (5.25b)$$

$$\nabla . \vec{D} = \rho \qquad (5.25c)$$

$$\nabla . \vec{B} = 0 \qquad (5.25d)$$

is known as the Maxwell's equation and this set of equations apply in the time varying scenario, static fields are being a particular case $\left(\frac{\partial}{\partial t} = 0\right)$.

In the integral form

$$\oint_c \vec{E}.d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t}.d\vec{S} \qquad (5.26a)$$

$$\oint_c \vec{H}.d\vec{l} = \int_s \left(J + \frac{\partial D}{\partial t}\right) d\vec{S} = I + \int_s \frac{\partial \vec{D}}{\partial t}.d\vec{S} \qquad (5.26b)$$

$$\int_V \nabla . \vec{D} dv = \oint_s \vec{D}.d\vec{S} = \int_V \rho dv \qquad (5.26c)$$

$$\oint \vec{B}.d\vec{S} = 0 \qquad (5.26d)$$

The modification of Ampere's law by Maxwell has led to the development of a unified electromagnetic field theory. By introducing the displacement current term, Maxwell could predict the propagation of EM waves. Existence of EM waves was later demonstrated by Hertz experimentally which led to the new era of radio communication.

5.5 Boundary Conditions for Electromagnetic fields

The differential forms of Maxwell's equations are used to solve for the field vectors provided the field quantities are single valued, bounded and continuous. At the media boundaries, the field vectors are discontinuous and their behaviors across the boundaries are governed by boundary conditions. The integral equations (eqn 5.26) are assumed to hold for regions containing discontinuous media. Boundary conditions can be derived by applying the Maxwell's equations in the integral form to small regions at the interface of the two media. The procedure is similar to those used for obtaining boundary conditions for static electric fields (chapter 2) and static magnetic fields (chapter 4). The boundary conditions are summarized as follows

With reference to fig 5.3

With reference to fig 5.3

$$\hat{a}_n \times \left(\vec{E}_1 - \vec{E}_2\right) = 0 \qquad 5.27(a)$$

$$\hat{a}_n \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_S \qquad 5.27(b)$$

$$\hat{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S \qquad 5.27(c)$$

$$\hat{a}_n \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \qquad 5.27(d)$$

Fig 5.4

Equation 5.27 (a) says that tangential component of electric field is continuous across the interface while from 5.27 (c) we note that tangential component of the magnetic field is discontinuous by an amount equal to the surface current density. Similarly 5.27 (b) states that normal component of electric flux density vector D(Bar) is discontinuous across the interface by an amount equal to the surface current density while normal component of the magnetic flux density is continuous. If one side of the interface, as shown in fig 5.4, is a perfect electric conductor, say region 2, a surface current $\vec{J_s}$ can exist even though $\vec{E}$ is zero as $\sigma = \infty$.

Thus eqn 5.27(a) and (c) reduces to

$$\hat{a}_n \times \vec{H} = \vec{J_s} \qquad (5.28(a))$$

$$\hat{a}_n \times \vec{E} = 0 \qquad (5.28(b))$$

5.6 Wave equation and their solution:

From equation 5.25 we can write the Maxwell's equations in the differential form as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{D} = \vec{\rho}$$

$$\nabla \cdot \vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant ε, magnetic permeability μ and conductivity σ. The above set of equations can be written as

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad (5.29(a))$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad (5.29(b))$$

$$\nabla \cdot \vec{E} = 0 \qquad (5.29(c))$$

$$\nabla \cdot \vec{H} = 0 \qquad (5.29(d))$$

Using the vector identity ,

$$\nabla \times \nabla \times \vec{A} = \nabla \cdot \left(\nabla \cdot \vec{A}\right) - \nabla^2 A$$

We can write from 5.29(b)

$$\nabla\times\nabla\times\vec{E} = \nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E}$$
$$= -\nabla\times\left(\mu\frac{\partial\vec{H}}{\partial t}\right)$$

or $$\nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\nabla\times\vec{H}\right)$$

Substituting $\nabla\times\vec{H}$ from 5.29(a)

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free medium $\nabla\cdot\vec{E} = 0$ (eqn 5.29(c))

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad (5.30)$$

In the same manner for equation eqn 5.29(a)

$$\nabla\times\nabla\times\vec{H} = \nabla\cdot\left(\nabla\cdot\vec{H}\right) - \nabla^2\vec{H}$$
$$= \sigma\left(\nabla\times\vec{E}\right) + \varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$
$$= \sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Since $\nabla\cdot\vec{H} = 0$ from eqn 5.29(d), we can write

$$\nabla^2\vec{H} = \mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right) + \mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right) \qquad (5.31)$$

These two equations

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu\sigma\left(\frac{\partial \vec{H}}{\partial t}\right) + \mu\varepsilon\left(\frac{\partial^2 \vec{H}}{\partial t^2}\right)$$

are known as wave equations.

It may be noted that the field components are functions of both space and time. For example, if we consider a Cartesian co ordinate system, $\vec{E}$ and $\vec{H}$ essentially represents $\vec{E}(x,y,z,t)$ and $\vec{H}(x,y,z,t)$. For simplicity, we consider propagation in free space , i.e. $\sigma = 0$, $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$. The wave eqn in equations 5.30 and 5.31 reduces to

$$\nabla^2 \vec{E} = \mu_0\varepsilon_0\left(\frac{\partial^2 \vec{E}}{\partial t^2}\right) \qquad (5.32(a))$$

$$\nabla^2 \vec{H} = \mu_0\varepsilon_0\left(\frac{\partial^2 \vec{H}}{\partial t^2}\right) \qquad (5.32(b))$$

Further simplifications can be made if we consider in Cartesian co ordinate system a special case where $\vec{E}$ and $\vec{H}$ are considered to be independent in two dimensions, say $\vec{E}$ and $\vec{H}$ are assumed to be independent of *y* and *z*. Such waves are called plane waves.

From eqn (5.32 (a)) we can write

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \varepsilon_0\mu_0\left(\frac{\partial^2 \vec{E}}{\partial t^2}\right)$$

The vector wave equation is equivalent to the three scalar equations

$$\frac{\partial^2 \vec{E}_x}{\partial x^2} = \varepsilon_0\mu_0\left(\frac{\partial^2 \vec{E}_x}{\partial t^2}\right) \qquad (5.33(a))$$

$$\frac{\partial^2 \vec{E}_y}{\partial x^2} = \varepsilon_0\mu_0\left(\frac{\partial^2 \vec{E}_y}{\partial t^2}\right) \qquad (5.33(b))$$

$$\frac{\partial^2 \overrightarrow{E_z}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \overrightarrow{E_z}}{\partial t^2} \right) \qquad (5.33(c))$$

Since we have $\nabla \cdot \vec{E} = 0$,

$$\therefore \frac{\partial \overrightarrow{E_x}}{\partial x} + \frac{\partial \overrightarrow{E_y}}{\partial y} + \frac{\partial \overrightarrow{E_z}}{\partial z} = 0 \qquad (5.34)$$

As we have assumed that the field components are independent of y and z eqn (5.34) reduces to

$$\frac{\partial E_x}{\partial x} = 0 \qquad (5.35)$$

i.e. there is no variation of E_x in the x direction.

Further, from 5.33(a), we find that $\frac{\partial E_x}{\partial x} = 0$ implies $\frac{\partial^2 E_x}{\partial t^2} = 0$ which requires any three of the conditions to be satisfied: (i) $E_x=0$, (ii)E_x = constant, (iii)E_x increasing uniformly with time.

A field component satisfying either of the last two conditions (i.e (ii) and (iii))is not a part of a plane wave motion and hence E_x is taken to be equal to zero. Therefore, a uniform plane wave propagating in x direction does not have a field component (*E* or *H*) acting along x.

Without loss of generality let us now consider a plane wave having $E_y$component only (Identical results can be obtained for E_z component) .

The equation involving such wave propagation is given by

$$\frac{\partial^2 \overrightarrow{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \overrightarrow{E_y}}{\partial t^2} \right) \qquad (5.36)$$

The above equation has a solution of the form

$$E_y = f_1(x - v_0 t) + f_2(x + v_0 t) \qquad (5.37)$$

where $v_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Thus equation (5.37) satisfies wave eqn (5.36) can be verified by substitution.

$f_1(x - v_0 t)$ corresponds to the wave traveling in the + x direction while $f_2(x + v_0 t)$ corresponds to a wave traveling in the -x direction. The general solution of the wave eqn thus consists of two waves, one traveling away from the source and other traveling back towards the source. In the absence of any reflection, the second form of the eqn (5.37) is zero and the solution can be written as

$$E_y = f_1(x - v_0 t) \qquad (5.38)$$

Such a wave motion is graphically shown in fig 5.5 at two instances of time t_1 and t_2.

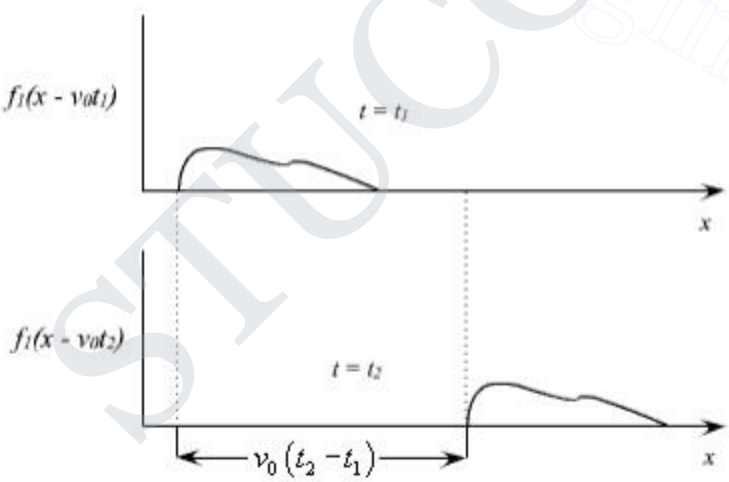


Fig 5.5 : Traveling wave in the + x direction

Let us now consider the relationship between E and H components for the forward traveling wave.

Since $\vec{E} = \widehat{a_y} E_y = \widehat{a_y} f_1(x - v_0 t)$ and there is no variation along y and z.

$$\nabla \times \vec{E} = \widehat{a_z} \frac{\partial E_y}{\partial x}$$

Since only z component of $\nabla \times \vec{E}$ exists, from (5.29(b))

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \qquad (5.39)$$

and from (5.29(a)) with $\sigma = 0$, only H_z component of magnetic field being present

$$\nabla \times \vec{H} = -\widehat{a_y} \frac{\partial H_z}{\partial x}$$

$$\therefore -\frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{\partial E_y}{\partial t} \qquad (5.40)$$

Substituting E_y from (5.38)

$$\frac{\partial H_z}{\partial x} = -\varepsilon_0 \frac{\partial E_y}{\partial t} = \varepsilon_0 v_0 f_1'(x - v_0 t)$$

$$\therefore \frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} f_1'(x - v_0 t)$$

$$\therefore H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \int f_1'(x - v_0 t) dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \frac{\partial}{\partial x} f_1 dx + c$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} f_1 + c$$

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y + c$$

The constant of integration means that a field independent of x may also exist. However, this field will not be a part of the wave motion.

Hence $H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y$ (5.41)

which relates the *E* and *H* components of the traveling wave.

$$z_0 = \frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi \text{ or } 377\Omega$$

$$z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

is called the characteristic or intrinsic impedance of the free space

5.7 Time Harmonic Fields

So far, in discussing time varying electromagnetic fields, we have considered arbitrary time dependence. The time dependence of the field quantities depends on the source functions. One of the most important case of time varying electromagnetic field is the time harmonic (sinusoidal or co sinusoidal) time variation where the excitation of the source varies sinusoidal in time with a single frequency. For time-harmonic fields, phasor analysis can be applied to obtain single frequency steady state response. Since Maxwell's equations are linear differential equations, for source functions with arbitrary time dependence, electromagnetic fields can be determined by superposition. Periodic time functions can be expanded into Fourier series of harmonic sinusoidal components while transient non-periodic functions can be expressed as Fourier integrals. Field vectors that vary with space coordinates and are sinusoidal function of time can be represented in terms of vector phasors that depend on the space coordinates but not on time. For time harmonic case, the general time variation is $e^{j\omega t}$ and for a cosine reference, the instantaneous fields can be written as:

$$\vec{E}(x,y,z,t) = \mathrm{Re}\left[\vec{E}(x,y,z)e^{j\omega t}\right] \qquad (5.42)$$

where $\vec{E}(x,y,z)$ is a vector phasor that contain the information on direction, magnitude and phase. The phasors in general are complex quantities. All time harmonic filed components can be written in this manner.

The time rate of change of $\vec{E}$ can be written as:

$$\frac{\partial \vec{E}(x,y,z,t)}{\partial t} = \mathrm{Re}\left[j\omega \vec{E}(x,y,z) e^{j\omega t} \right] \quad (5.43)$$

Thus we find that if the electric field vector $\vec{E}(x,y,z,t)$ is represented in the phasor form as $\vec{E}(x,y,z)$, then $\frac{\partial \vec{E}(x,y,z,t)}{\partial t}$ can be represented by the phasor $j\omega\vec{E}(x,y,z)$. The integral $\int \vec{E}(x,y,z,t)dt$ can be represented by the phasor $\frac{\vec{E}(x,y,z)}{j\omega}$. In the same manner, higher order derivatives and integrals with respect to t can be represented by multiplication and division of the phasor $\vec{E}(x,y,z)$ by higher power of $j\omega$. Considering the field phasors $(\vec{E},\vec{H})$ and source phasors $(\rho,\vec{J})$ in a simple linear isotropic medium, we can write the Maxwell's equations for time harmonic case in the phasor form as:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad (5.44a)$$

$$\nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E} \quad (5.44b)$$

$$\nabla . \vec{E} = \frac{\rho}{\omega} \quad (5.44c)$$

$$\nabla . \vec{H} = 0 \quad (5.44d)$$

Similarly, the wave equations described in equation (5.32) can be written as:

$$\nabla^2 \vec{E} = \mu_0\varepsilon_0 (j\omega)^2 \vec{E} = -\omega^2 \mu_0 \varepsilon_0 \vec{E}$$

or $\nabla^2 \vec{E} + k_0^2 \vec{E} = 0$ (5.45a)

And in the same manner, for the magnetic field

$$\nabla^2 \vec{H} + k_0^2 \vec{H} = 0$$

where $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ is called the wave number

UNIT V

PLANE ELECTROMAGNETIC WAVES

Plane EM Wave in a Lossy and Lossless Media:

$\nabla \times H = \vec{J} + j\omega\varepsilon\vec{E} = \sigma\vec{E} + j\omega\varepsilon\vec{E} = j\omega(\varepsilon - j\frac{\sigma}{\omega})\vec{E} = j\omega\varepsilon_c\vec{E}$, $\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon''$.

Similarly, $\mu_c = \mu' - j\mu''$

Complex wave number: $k_c = \omega\sqrt{\mu\varepsilon_c}$. **Loss tangent:** $\tan\delta_c \approx \varepsilon''/\varepsilon' = \frac{\sigma}{\omega\varepsilon}$

Propagation constant: $\gamma = jk_c = j\omega\sqrt{\mu\varepsilon_c} = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}(1+\frac{\sigma}{j\omega\varepsilon})^{1/2}$

$E \propto e^{-\gamma z} = e^{-jk_c z} = e^{-\alpha z}\cdot e^{-j\beta z}$

If the medium is lossless, α=0; else if the medium is lossy, α>0.

Phase constant: $\beta = \frac{2\pi}{\lambda}$

$$\Rightarrow \alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}}[\sqrt{1+(\frac{\sigma}{\omega\varepsilon})^2} - 1]^{1/2}, \ \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}}[\sqrt{1+(\frac{\sigma}{\omega\varepsilon})^2} + 1]^{1/2}$$

Case 1 Low-loss Dielectric: $\frac{\sigma}{\omega\varepsilon} << 1 \Rightarrow \alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$, $\beta \approx \omega\sqrt{\mu\varepsilon}[(1+\frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$

Intrinsic impedance: $\eta_c \approx \sqrt{\frac{\mu}{\varepsilon}}(1+j\frac{\sigma}{2\omega\varepsilon})$

Phase velocity: $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon_c}} \approx \frac{1}{\sqrt{\mu\varepsilon}}[1-\frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2]$

Case 2 Good Conductor: $\frac{\sigma}{\omega\varepsilon} >> 1 \Rightarrow \alpha = \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma}$,

and $\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \approx (\sqrt{\frac{j\omega\mu}{\sigma}}) = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}$

Phase velocity: $v_p = \frac{\omega}{\beta} \approx \sqrt{\frac{2\omega}{\mu\sigma}}$

Skin Depth (depth of penetration): $\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}}$.

For a good conductor, $\delta = \frac{1}{\alpha} \approx \frac{1}{\beta} = \frac{\lambda}{2\pi}$

Eg. $\vec{E}(t,z) = \hat{x}100\cos(10^7\pi t)$ ***V/m* at *z*=0 in seawater: ε_r=72, μ_r=1, σ=4*S/m*. (a) Determine *α*, *β*, v_p, and η_c. (b) Find the distance at which the amplitude of *E* is 1% of its value at *z*=0. (c) Write *E*(*z*,*t*) and *H*(*z*,*t*) at *z*=0.8*m*, suppose it propagates in the +*z* direction.**

(Sol.) $\omega = 10^7\pi$, f=5×10^{6}Hz, $\sigma/\omega\varepsilon_0\varepsilon_r$=200>>1, ∴ Seawater is a good conductor in this case.

(a) $\alpha = \sqrt{\pi f \mu \sigma} = 8.89 Np/m = \beta$, $\eta_c = (1+j)\sqrt{\dfrac{\pi f \mu}{\sigma}}$

$$v_p = \frac{\omega}{\beta} = 3.53\times10^6 m/s,\ \lambda = \frac{2\pi}{\beta} = 0.707m,\ \delta = \frac{1}{\alpha} = 0.112m$$

(b) $e^{-\alpha z} = 0.01 \Rightarrow z = \dfrac{1}{\alpha}\ln(100) = 0.518m$

(c) $E(z,t) = \mathrm{Re}[E(z)e^{j\omega t}] = \hat{x}100e^{-\alpha z}\cos(\omega t - \beta z)$

$$z = 0.8m \Rightarrow E(0.8,t) = \hat{x}100e^{-0.8\alpha}\cos(\omega t - 0.8\beta) = \hat{x}0.082\cos(10^7\pi t - 7.11)$$

$$\vec{H}(0.8,t) = \frac{1}{\eta}\hat{a}_n \times \vec{E}(0.8,t),\ H(0.8,t) = \hat{y}\,\mathrm{Re}[\frac{E_x(0.8)}{\eta_c}e^{j\omega t}] = \hat{y}0.026\cos(10^7\pi t - 1.61)$$

Eg. The magnetic field intensity of a linearly polarized uniform plane wave propagating in the +*y* direction in seawater ε_r=80, μ_r=1, σ=4*S/m* is $\vec{H} = \hat{x}0.1\sin(10^{10}\pi t - \frac{\pi}{3})$ *A/m*. (a) Determine the attenuation constant, the phase constant, the intrinsic impedance, the phase velocity, the wavelength, and the skin depth. (b) Find the location at which the amplitude of *H* is 0.01 *A/m*. (c) Write the expressions for *E*(*y*,*t*) and *H*(*y*,*t*) at *y*=0.5*m* as function of *t*.

(Sol.) (a) $\sigma/\omega\varepsilon$=0.18<<1, ∴ Seawater is a low-loss dielectric in this case.

$$\Rightarrow \alpha \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = 83.96 Np/m \quad \eta_c \approx \sqrt{\frac{\mu}{\varepsilon}}(1 + j\frac{\sigma}{2\omega\varepsilon}) = 41.8e^{j0.0283\pi}$$

$$\beta \approx \omega\sqrt{\mu\varepsilon}[(1 + \frac{1}{8}(\frac{\sigma}{\omega\varepsilon})^2] = 300\pi,\quad v_p = \frac{\omega}{\beta} = 3.33\times10^7 m/s,\quad \delta = \frac{1}{\alpha} = 1.19\times10^{-2}m,$$

$$\lambda = \frac{2\pi}{\beta} = 6.67\times10^{-3}m$$

(b) $e^{-\alpha y} = \dfrac{0.01}{0.1} \Rightarrow y = \dfrac{1}{\alpha}\ln 10 = 2.74\times10^{-2}m$

(c) $H(y,t) = \hat{x}0.1e^{-\alpha y}\sin(10^{10}\pi t - \beta y - \dfrac{\pi}{3})$, $y = 0.5, \beta = 300\pi$

$$\Rightarrow \vec{H}(0.5,t) = \hat{x}5.75\times10^{-20}\sin(10^{10}\pi t - \frac{\pi}{3})$$

$$\hat{a}_n = \hat{y} \Rightarrow \vec{E}(0.5,t) = -\eta_c\hat{a}_n \times \vec{H}(0.5,t) = \hat{z}2.41\times10^{-18}\sin(10^{10}\pi t - \frac{\pi}{3} + 0.0283\pi)$$

Eg. Given that the skin depth for graphite at 100 *MHz* is 0.16*mm*, determine (a) the conductivity of graphite, and (b) the distance that a 1*GHz* wave travels in graphite such that its field intensity is reduced by 30*dB*.

(Sol.) (a) $\delta = \dfrac{1}{\sqrt{\pi f \mu \sigma}} = 0.16\times10^{-3} \Rightarrow \sigma = 0.99\times10^{5}\,S/m$

(b) At f=$10^9 Hz$, $\alpha = \sqrt{\pi f\mu\sigma} = 1.98\times10^4\,Np/m$

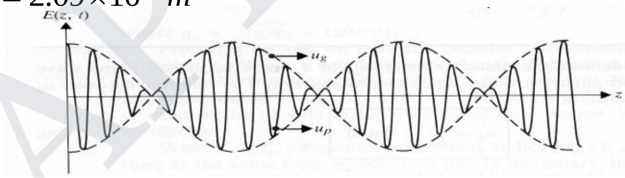
$$-30(dB) = 20\log_{10} e^{-\alpha z} \Rightarrow z = \frac{1.5}{\alpha\log_{10} e} = 1.75\times10^{-4}\,m$$

Eg. Determine and compare the intrinsic impedance, attenuation constant, and skin depth of copper σ_{cu}=$5.8\times10^7 S/m$, silver σ_{ag}=$6.15\times10^7 S/m$, and brass σ_{br}=$1.59\times10^7 S/m$ at following frequencies: 60*Hz* and 1*GHz*.

(Sol.) $\alpha = \sqrt{\pi f\mu\sigma}$, $\delta = \dfrac{1}{\alpha}$, $f\uparrow \Rightarrow \delta\downarrow$, $\eta_c = (1+j)\dfrac{\alpha}{\sigma}$

Copper: $60Hz \Rightarrow \eta_c = 2.02(1+j)\times10^{-6}\,\Omega$, $\alpha = 1.17\times10^2\,Np/m$, $\delta = 8.53\times10^{-3}\,m$

$1GHz \Rightarrow \eta_c = 8.25(1+j)\times10^{-3}\,\Omega$, $\alpha = 4.79\times10^5\,Np/m$, $\delta = 2.09\times10^{-6}\,m$



Group velocity:

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{d\beta/d\omega}$$

$$\vec{E}(t,z) = E_0\cos[(\omega+\Delta\omega)t - (\beta+\Delta\beta)z] + E_0\cos[(\omega-\Delta\omega)t - (\beta-\Delta\beta)z]$$
$$= 2E_0\cos(t\Delta\omega - z\Delta\beta)\cos(\omega t - \beta z)$$

Let $t\Delta\omega - z\Delta\beta$ =constant $\Rightarrow v_g = \dfrac{dz}{dt} = \dfrac{\Delta\omega}{\Delta\beta} = \dfrac{1}{\Delta\beta/\Delta\omega} = \dfrac{d\omega}{d\beta} = \dfrac{1}{d\beta/d\omega}$

Eg. Show that $v_g = v_p + \beta\dfrac{dv_p}{d\beta}$ **and** $v_g = v_p - \lambda\dfrac{dv_p}{d\lambda}$

(Proof) $v_p = \dfrac{\omega}{\beta}$, $\omega = v_p\beta$, $v_g = \dfrac{d\omega}{d\beta} = v_p + \beta\dfrac{dv_p}{d\beta}$

$\because\ \beta = \dfrac{2\pi}{\lambda}$, $\beta\lambda = 2\pi$, $\lambda d\beta + \beta d\lambda = 0 \Rightarrow \dfrac{\beta}{d\beta} = -\dfrac{\lambda}{d\lambda}$, $v_g = v_p - \lambda\dfrac{dv_p}{d\lambda}$

Poynting vector and Theorem:

$$\vec{P} = \vec{E}\times\vec{H}$$

$\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$, $\nabla\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t} \Rightarrow \nabla\cdot(\vec{E}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H}) = -\vec{H}\cdot\dfrac{\partial\vec{B}}{\partial t} - \vec{E}\cdot\dfrac{\partial\vec{D}}{\partial t} - \vec{E}\cdot\vec{J}$

$$= -\vec{H}\cdot\frac{\partial(\mu\vec{H})}{\partial t} - \vec{E}\cdot\frac{\partial(\varepsilon\vec{E})}{\partial t} - \vec{E}\cdot\vec{J} = -\frac{\partial}{\partial t}(\frac{1}{2}\mu|\vec{H}|^2) - \frac{\partial}{\partial t}(\frac{1}{2}\varepsilon|\vec{E}|^2) - \sigma|\vec{E}|^2$$

$$\therefore \oiint_s (\vec{E}\times\vec{H})\cdot d\vec{S} = \iiint_v \nabla\cdot(\vec{E}\times\vec{H})dv = -\frac{\partial}{\partial t}\iiint_v (\frac{\varepsilon}{2}|\vec{E}|^2 + \frac{\mu}{2}|\vec{H}|^2)dv - \iiint_v \sigma|\vec{E}|^2 dv$$

$\Rightarrow \vec{P} = \vec{E}\times\vec{H}$ is the electromagnetic power flow per unit area.

Instantaneous power density: $\vec{P}(z,t) = \mathrm{Re}[\vec{E}(z)e^{j\omega t}] \times \mathrm{Re}[\vec{H}(z)e^{j\omega t}]$

Set $\vec{E}(z) = \hat{x}E_x(z) = \hat{x}E_0 e^{-(\alpha + j\beta)z} \Rightarrow \vec{H}(z) = \frac{1}{\eta}[\hat{a}_n \times \vec{E}(z)] = \hat{y}\frac{E_0}{|\eta|}e^{-\alpha z}\cdot e^{-j(\beta z + \theta_\eta)}$,

$\therefore\ \vec{E}(z,t) = \mathrm{Re}[\vec{E}(z)e^{j\omega t}] = \hat{x}E_0 e^{-\alpha z}\cos(\omega t - \beta z)$

and $\vec{H}(z,t) = \mathrm{Re}[H(z)e^{j\omega t}] = \hat{y}\frac{E_0}{|\eta|}e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)$

$$\Rightarrow \vec{P}(z,t) = \vec{E}(z,t) \times \vec{H}(z,t) = \mathrm{Re}[\vec{E}(z)e^{j\omega t}] \times \mathrm{Re}[\vec{H}(z)e^{j\omega t}]$$

$$= \hat{z}\frac{|E_0|^2}{2|\eta|}e^{-2\alpha z}[\cos\theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)] \propto |E_0|^2$$

Average power density: $\vec{P}_{av} = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*)$

$\vec{P}_{av} = \frac{1}{T}\int_0^T \vec{P}(z,t)dt = \hat{z}\frac{|E_0|^2}{2|\eta|}e^{-2\alpha z}\cos\theta_\eta$, where T is the period. And it can be proved that $\vec{P}_{av} = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*)$.

Eg. Show that $\vec{P}(z,t)$ of a circularly–polarized plane wave propagating in a lossless medium is a constant.

(Sol.) Assuming right–hand circularly–polarized plane wave, $\hat{a}_n = \hat{z}$

$$\vec{E}(z,t) = E_0[\hat{x}\cos(\omega t - \beta z) + \hat{y}\sin(\omega t - \beta z)]$$

$$\vec{H}(z,t) = \frac{1}{\eta}(\hat{a}_n \times \vec{E}) = \frac{E_0}{\eta}[-\hat{x}\sin(\omega t - \beta z) + \hat{y}\cos(\omega t - \beta z)]$$

$$\vec{P}(z,t) = \vec{E}(z,t) \times \vec{H}(z,t) = \hat{z}\frac{E_0{}^2}{\eta}$$

Eg. The radiation electric field intensity of an antenna system is $\vec{E} = \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi$, find the expression for the average outward power flow per unit area.

(Sol.) $\hat{a}_n = \hat{a}_r$, $\vec{H} = \frac{1}{\eta}(\hat{a}_n \times \vec{E}) = (-\hat{a}_\theta\frac{E_\phi}{\eta} + \hat{a}_\phi\frac{E_\theta}{\eta})$

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}(\vec{E} \times \vec{H}^*) = \frac{1}{2}\mathrm{Re}[(\hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) \times (-\hat{a}_\theta\frac{E_\phi{}^*}{\eta} + \hat{a}_\phi\frac{E_\theta{}^*}{\eta})] = \frac{1}{2\eta}\hat{a}_r(|E_\theta|^2 + |E_\phi|^2)$$

Eg. Find $\vec{P}$ on the surface of a long, straight conducting wire of radius *b* and conductivity σ that carries a direct current *I*. Verify Poynting's theorem.

(Sol.) $\vec{J} = \hat{z}\frac{I}{\pi b^2} \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma} = \hat{z}\frac{I}{\sigma\pi b^2}$, $\vec{H} = \hat{a}_\phi\frac{I}{2\pi b} \Rightarrow \vec{P} = \vec{E} \times \vec{H} = -\hat{a}_r\frac{I^2}{2\sigma\pi^2 b^3}$

$$-\oiint_s \vec{P}\cdot d\vec{S} = -\oiint_s \vec{P}\cdot\hat{a}_r dS = \frac{I^2}{2\sigma\pi^2 b^2}\cdot 2\pi b\ell = I^2(\frac{\ell}{\sigma\pi b^2}) = I^2 R$$

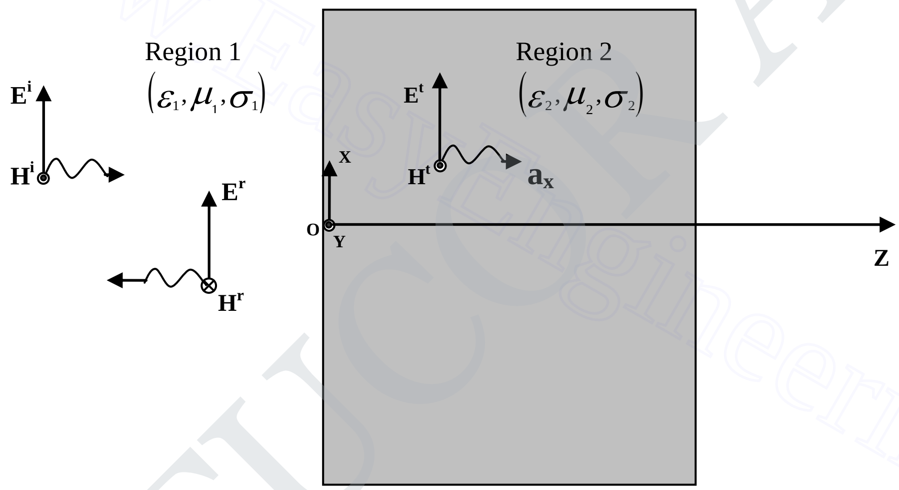
Normal Incidence Plane Wave Reflection and Transmission at Plane at Plane Boundaries

Introduction

Why is there a need to study the reflection and transmission properties of plane waves when incident on boundaries between regions of different electric properties? Perhaps you had no idea that we experience this topic daily in our lives. For instance, when you try and make a call on your cell phone and you are downtown amongst all those tall buildings. Will you always have great reception? When the hot sun penetrates your window it can quickly heat up your room, but maybe you have blinds, curtains, or tinted material to prevent some of that intense heat. For those of you that wear glasses, you know what happens when you get your picture taken; that annoying glare from those glasses. What about a light ray on the surface of a mirror? A reflection can be seen and some of that ray will penetrate the glass.

This chapter focuses on the reflection and transmission properties related to one-dimensional problems that have normal-incident plane waves converging on infinite plane interfaces that will separate two or more different media.

The Figure illustrates the geometry of the positive z propagating plane wave that is normally incident on a plane interface between regions 1 & 2.



Normal Incidence Plane Wave Reflection and Transmissions at Plane Boundary Between Two Conductive Media

The electric and magnetic fields related to the incident wave are given by the following:

$$\hat{E}_x^i = \hat{E}_{m1}^+ e^{-\hat{\gamma}_1 z}$$

$$\hat{H}_y^i = \frac{\hat{E}_{m1}^+}{\hat{\eta}_1} e^{-\hat{\gamma}_1 z}$$

* Note: (i) incident, (m_1) medium 1, (γ_1) propagation constant in region 1, (η1) wave impedance in region 1, (z) direction of propagating wave

The complex propagation constant in region 1 is $\hat{\gamma} = \alpha_1 + j\beta_1$. *Note: α and β are the real and imaginary parts respectively. The propagation constant γ is that square root of γ^2 whose real and imaginary parts are positive:

$$\gamma = \alpha + j\beta$$

With
$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right)}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right)}$$

The wave impedance as defined in chapter 2 as the ratio between the electric and magnetic fields is

$$\frac{\hat{E}_x}{\hat{H}_y} = \hat{\eta} = \frac{\mu}{\left(\varepsilon - j\dfrac{\sigma}{\omega}\right)} = \frac{\sqrt{\dfrac{\mu}{\varepsilon}}}{\left[1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}} e^{j\frac{1}{2}\tan^{-1}\left(\frac{\sigma}{\omega\varepsilon}\right)}$$

The wave impedance η in a conductive medium is a complex number meaning that the electric and magnetic fields are not in phase. The phase velocity will be less than the velocity of light $v_p < c$. The wavelength λ in the conductive medium will be shorter than the wavelength λ_o in free space at the same frequency, $\lambda = 2\pi/\beta < \lambda_o$. The factor $e^{-\alpha z}$ will attenuate the magnitudes of both E and H as they propagate in the +z direction.

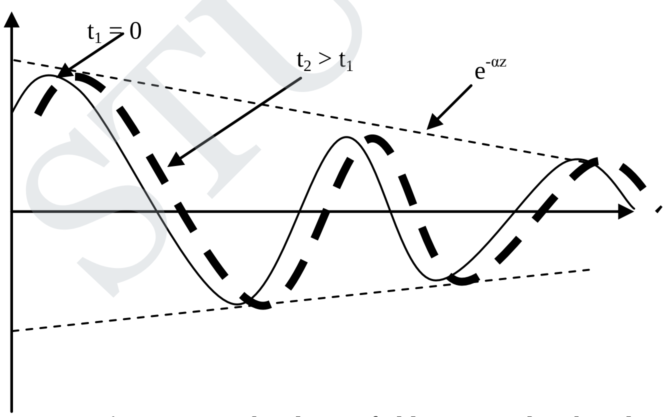


Figure 5.2 The electric field associated with a plane wave propagating along the positive z direction.

What happens when this wave hits the boundary?

Some of the energy related to the incident wave will transmit across the boundary surface at z = 0 in region 2, therefore providing a transmitted wave in the +z direction in medium 2. The following are the electric and magnetic fields related to the transmitted wave:

$$\hat{E}_x^t = \hat{E}_{m2}^+ e^{-\hat{\gamma}_2 z}$$

$$\hat{H}_y^t = \frac{\hat{E}_{m2}^+}{\hat{\eta}_2} e^{-\hat{\gamma}_2 z}$$

* Note: (t) transmitted wave

Recall Maxwell's equations:

$$\nabla \wedge H = (\varepsilon + j\sigma)E$$
$$\nabla \wedge E = jB$$
$$\nabla \cdot E = 0$$
$$\nabla \cdot H = 0$$

The Wave equation for H:

$$\frac{\partial^2 H}{\partial^2 x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = \gamma^2 H$$

For now, let's look at the simplest system, that consisting of a plane wave of coordinate z.

$$\frac{d^2 H}{d z^2} = \gamma^2 H$$

Therefore, according to the wave equation as noted above, equations satisfy Maxwell's equations. If the amplitude of the transmitted wave $\hat{E}_{m2}^+$ is unknown then boundary conditions at the interface z = 0 separating the two media must be satisfied.
Good conductors are often treated as if they were perfect conductors. Metallic conductors such as copper have a high conductivity $\sigma = 6 * 10^7$ S/m, however, only superconductors have infinite conductivity and are truly perfect conductors.

- Static (time independent)

$$n \cdot D_1 = \rho_s$$
$$n \cdot (B_1 - B_2) = 0$$
$$n \wedge E_1 = 0$$
$$n \wedge (H_1 - H_2) = 0$$

* Subscripts denote the conducting medium.

Characteristics of static cases:

1. Electrostatic field inside a good conducting medium is zero. Free charge can exist on the surface of a conductor, thus making the normal component of **D** discontinuous being zero inside the conductor and nonzero outside. The

tangential component of **E** just inside the conductor must be zero even if the surface is charged.

2. The electric and magnetic fields in the static case are independent. A static magnetic field can therefore exist inside a metallic body, even though an **E** field cannot. The normal component of **B** and the tangential components of **H** are therefore continuous across the interface.

For time-varying fields, the boundary conditions for good (perfect) conductors are:

- Time –varying fields (time dependent)

$$n \cdot D = 0$$
$$n \cdot B = 0$$
$$n \wedge E = 0$$
$$n \wedge H = J_s$$

The subscripts have been deleted because in this case the only nonvanishing fields are those outside the conducting body.

$\hat{E}_x^t$ & $\hat{E}_x^i$ are tangential to the interface therefore the boundary conditions will require these fields be equal at z = 0. Equate $\hat{E}_x^t$ & $\hat{E}_x^i$, and set z = 0. *Note: (t) is transmitted wave, (i) incident wave. The result will be

$$\hat{E}_{m1}^{+} = \hat{E}_{m2}^{+} \quad \text{*Note: (m) medium, (+) transmitted wave} \qquad (5.3)$$

$\hat{H}_y^t$ & $\hat{H}_y^i$ are also tangential to the interface, so by applying the same procedure as above you will notice that it is impossible to satisfy the magnetic field boundary conditions if $\hat{\eta}_1 \neq \hat{\eta}_2$.

We can then, include a reflected wave in region 1 traveling away from the interface, or in other words in the –z direction. Only part of the energy related to the incident wave will be transmitted to region 2 because of the process the incident fields must encounter prior to crossing the boundary. The fields left behind during this process will in fact be the **reflected wave**.

The electric and magnetic fields related to the reflected wave are

$$\hat{E}_x^r = \hat{E}_{m1}^{-} e^{-\hat{\gamma}_1 z}$$

$$(5.4)$$

$$\hat{H}_y^r = \frac{-\hat{E}_{m1}^{-}}{\hat{\eta}_1} e^{-\hat{\gamma}_1 z}$$

Note: (r) reflected wave, (-) wave traveling in the –z direction

Equation (5.4) is related by $\dfrac{\hat{E}_x^r}{\hat{H}_y^r} = -\hat{\eta}_1$ because the reflected wave is traveling in the –z direction and the Poynting vector $\mathbf{E} \wedge \mathbf{H}$ will be in the -a_z direction. To satisfy the boundary conditions for the tangential electric field, z = 0. This is important because the basic model assumes three waves $\Rightarrow$ incident, reflected and transmitted:

$$\left(\hat{E}_x^i + \hat{E}_x^r\right)\Big|_{z=0} = \hat{E}_x^t\Big|_{z=0}$$

This can be simplified, by adding the E field transmitted to the E field reflected of medium 1 with a result equal to the E field transmitted wave of medium 2:

$$\hat{E}_{m1}^+ + \hat{E}_{m1}^- = \hat{E}_{m2}^+ \qquad (5.5)$$

*Note: We can model the system as three waves $\Rightarrow$ incident, reflected and transmitted. Boundary conditions must be met for the **E** field as well as the **H** field. Waves have both **E** & **H** fields - $\vec{\omega} * E = \eta * H$.

Similarly, enforcing the continuity of the tangential magnetic field at z = 0,

$$\hat{H}_{m1}^+ + \hat{H}_{m1}^- = \hat{H}_{m2}^+$$

Therefore,

$$\frac{\hat{E}_{m1}^+}{\hat{\eta}_1} - \frac{\hat{E}_{m1}^-}{\hat{\eta}_1} = \frac{\hat{E}_{m2}^+}{\hat{\eta}_2} \qquad (5.6)$$

To solve for $\hat{E}_{m2}^+$, multiply equation (5.6) by $\hat{\eta}_1$ and add the result to equation (5.5). The result is:

$$\frac{\hat{E}_{m2}^+}{\hat{E}_{m1}^+} = \frac{2\hat{\eta}_2}{\hat{\eta}_1 + \hat{\eta}_2} = \hat{T} \qquad (5.7)$$

The transmission coefficient $\hat{T}$ is the ratio of the amplitudes of the transmitted to the incident fields:

$$\hat{T} = \frac{2\hat{\eta}_2}{\hat{\eta}_1 + \hat{\eta}_2} \qquad (5.8)$$

The amplitude of the reflected wave can be solved for by multiplying equation (5.6) by $\hat{\eta}_2$ and subtracting the result from equation (5.5) for a result of:

$$\hat{E}_{m1}^- = \hat{E}_{m1}^+ \frac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1} \qquad (5.9)$$

The reflection coefficient $\hat{\Gamma}$ is the ratio of the amplitudes of the reflected and incident electric fields given by:

$$\hat{\Gamma} = \frac{\hat{E}_{m1}^-}{\hat{E}_{m1}^+} = \frac{\hat{\eta}_2 - \hat{\eta}_1}{\hat{\eta}_2 + \hat{\eta}_1} \qquad (5.10)$$

From equations (5.8) & (5.10), note that the reflection and transmission coefficients are related by $1+\hat{\Gamma}=\hat{\mathrm{T}}$.

EXAMPLES:

An **H** field travels in the $-a_z$ direction in free space with a phaseshift constant (β) of 30.0 rad/m and an amplitude of $\frac{1}{3\pi}$ A/m. If the field has the direction $-\mathbf{a}_y$ when t = 0 and z = 0, write suitable expressions for **E** and **H**. Determine the frequency and wavelength.

In a medium of conductivity σ, the intrinsic impedance η, which relates **E** and **H**, would be complex, and so the phase of **E** and **H** would have to be written in complex form. In free space the restriction is unnecessary. Using cosines, then

$$\mathbf{H}(z,t) = -\frac{1}{3\pi}\cos(\omega t+\beta z)$$

For propagation on –z,

$$\frac{E_x}{H_y}=-\eta_o=-120\pi\ \Omega \qquad \text{Or} \qquad E_x=+40\cos(\omega t+\beta z)\ \left(V/m\right)$$

Thus $$E(z,t)=40\cos(\varpi t+\beta z)(V/m)$$

Since $\beta=30\,rad/m$,

$$\lambda=\frac{2\pi}{\beta}=\frac{\pi}{15}m \qquad f=\frac{c}{\lambda}=\frac{3*10^8}{\pi/15}=\frac{45}{\pi}*10^8\ \text{Hz}$$

Determine the propagation constant γ for a material having $\mu_r=1, \varepsilon_r=8$, and $\sigma=0.25\rho S/m$, if the wave frequency is 1.6MHz.

In this case,

$$\frac{\sigma}{\omega\varepsilon}=\frac{0.25*10^{-12}}{2\pi\left(1.6*10^6\right)(8)\left(10^{-9}/36\pi\right)}\approx10^{-9}\approx0$$

So that

$$\alpha=0 \qquad \beta\approx\omega\sqrt{\mu\varepsilon}=2\pi f\frac{\sqrt{\mu_r\varepsilon_r}}{c}=9.48*10^{-2}\ rad/m$$

And $\gamma=\alpha+j\beta\approx j9.48*10^{-2}\ m^{-1}$. The material behaves like a perfect dielectric at the given frequency. Conductivity of the order $1\rho S/m$ indicates that the material is more like an insulator than a conductor.

5.3 Normal Incidence Plane-Wave Reflection at Perfectly conducting Plane

Special case (analysis of material presented in section 5.2)

Assumptions-(region 2) perfect conductor $\sigma^2 \rightarrow \infty$, wave impedance

$$\hat{\eta}_2 = \sqrt{\frac{\mu_2}{\varepsilon_2 - j\dfrac{\sigma_2}{\omega}}} = 0 \text{, as } \sigma_2 \rightarrow \infty, \qquad (5.11)$$

To simplify the standing wave analysis, assume that region 1 is a perfect dielectric $\sigma_1 = 0$.

Using substitution: take equation (5.11) in the reflection and transmission coefficient expressions in equations (5.8) and (5.10) in order to obtain

$$\hat{T} = 0, \qquad \hat{\Gamma} = -1$$

The zero value of the transmission coefficient simply means that the amplitude of the transmitted field in region 2 is $\hat{E}^+_{m2} = 0$. This can be explained in terms of the following:

- The depth of penetration parameter is zero in a perfectly conducting region, (Chapter 3, p. 241). Therefore, there will be no transmitted wave in a perfectly conducting region, because of the inability of time-varying fields to penetrate media with conductivities converging toward infinity.
- Only the incident and reflected fields will be present in region 1.

For $\hat{\Gamma} = -1$,

- The amplitude of the reflected wave is $\hat{E}^-_{m1} = -\hat{E}^+_{m1}$. The reflected wave is therefore equal in amplitude and is opposite in phase to the incident wave. This simply means that the entire incident energy wave is reflected back by the perfect conductor.
- The combination of the two fields meets the boundary condition at the surface of the perfect conductor.

This can be illustrated by examining the expression for the total electric field $E^{tot}(z)$ in region 1, which is assumed to be a perfect dielectric (i.e., $\alpha_1 = 0$)

$$\hat{E}^{tot}(z) = \hat{E}^i(z) + \hat{E}^r(z) = \hat{E}^+_{m1} e^{-j\beta_1 z} a_x + \hat{E}^-_{m1} e^{-j\beta_1 z} a_x$$

Substituting $\hat{E}^-_{m1} = -\hat{E}^+_{m1}$, for a result of:

$$\hat{E}^{tot}(z) = \hat{E}^+_{m1}\left(e^{-j\beta_1 z} - e^{j\beta_1 z}\right) a_x$$

$$= \text{-2j} - 2j\,\hat{E}^+_{m1} \sin(\beta_1 z) \sin \omega t\, a_x \qquad (5.12)$$

Note: The total electric field is zero at the perfectly conducting surface (z = 0) meeting the boundary condition.

To study the propagation characteristics of the compound wave in front of the perfect conductor, we must obtain the real-time form of the electric field.

Step 1: Multiply the complex form of the field in equation (5.12) by $e^{j\omega t}$
Step 2: Take the real part of the resulting expression

$$\hat{E}^{tot}(z) = \mathrm{Re}\, e^{jwt}\left[\hat{E}^{tot}(z)\right]$$
$$= 2\,{}^{+}_{m1} \sin(\beta_1 z) \sin \omega t \; a_x \qquad (5.13)$$

In equation (5.13) the amplitude of the electric field was assumed real $\hat{E}^{+}_{m1}$. Our objective is then to complete the following step:

Step 3: Show that the total field in region 1 is not a traveling wave, although it was obtained by combining two traveling waves of the same frequency and equal amplitudes of which are propagating in the opposite direction.

Figure 5.2 shows a variation of the total electric field in equation (5.13) as a function of z at various time intervals.

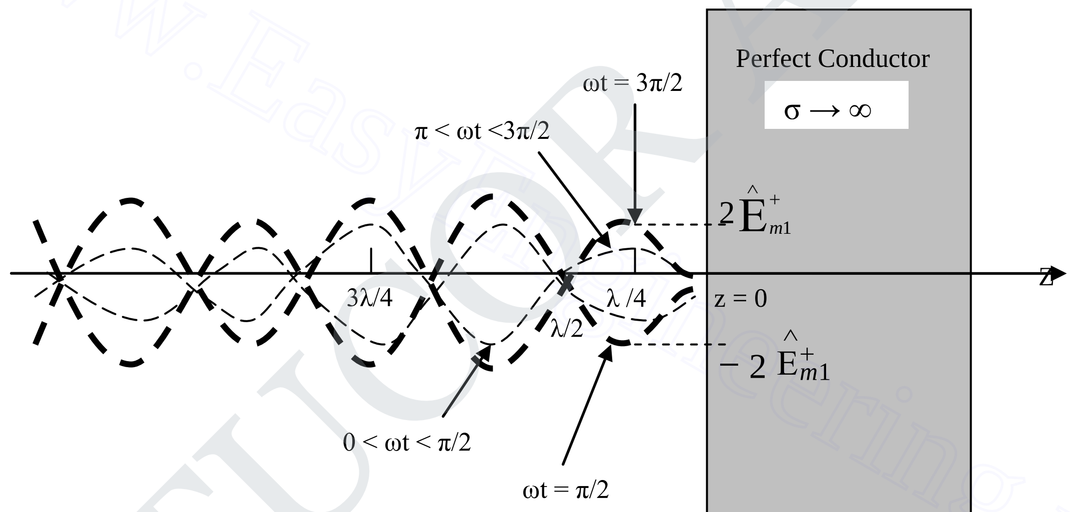


Figure 5.2b The variation of the total electric field in front of the perfect conductor as a function of **z** and at various time intervals ωt.

From figure 5.2 you can make the following observations:

1. α = 0, indicating that the total field meets the boundary condition at all times.

2. The total electric field has maximum amplitude twice that of the incident wave. The maximum amplitude occurs at z = λ/4, at z = 3λ/4, etc., when ωt = π/2, ωt = 3π/2, etc. happening when both the incident and reflected waves constructively interfere.

3. When z = λ/2, z = λ, z = 3λ/2, etc., in front of the perfect conductor the total electric field is always zero. This is happening when the two fields are going through destructive interference process for all values of ωt, also known as null locations.

4. The occurrence of the null and constructive interference locations do not change with time. The only thing that changes with time is the amplitude of the total field at nonnull locations. Therefore, the wave resulting from the interference of the two waves is called "standing waves".

It should also be emphasized that the difference between the electric field expressions for the traveling and standing waves. For a traveling wave, the electric field is given by:

$$E(z,t) = E_{m1}^{+} \cos(\omega t - \beta_1) a_x$$

The term $(\omega t - \beta_1 z)$ *or* $\omega\,(t - z/v_1)$ emphasizes the coupling between the location as a function of time of a specific point (constant phase) propagating along the wave.
It also indicates with an increase in t, z should also increase in order to maintain a constant value of ($t - z/v_1$), and it characterizes a specific point on the wave. This means that a wave with an electric field expression which includes $\cos(\omega t - \beta_1 z)$ is a propagating wave in the positive z direction. The time t and location z variables are uncoupled in equation (5.13), or in other words, the electric field distribution as a function of z in front of the perfect conductor follows a $\sin(\beta_1 z)$ variation, with the locations of the field nulls being those values of z at which $\sin(\beta_1 z) = 0$.
The sin (ωt) term modifies the amplitude of the field allowing a variation of a function of time located at the nonzero field locations as illustrated in Figure 5.2.
By finding the values of $(\beta_1 z)$ the permanent locations of the electric field nulls can be determined, thus making the value of the field zero. So, from equation (5.12) we can see that

$$\hat{E}^{tot}(z) = 0 \ at\ \beta_1\ z = n\pi \quad (n = 0, \pm 1, \pm 2, ...$$

Therefore, $$z = \frac{2\pi}{\lambda_1} z = n\pi$$

Or $$z = n\frac{\lambda_1}{2} \qquad (5.14)$$

This simply shows that $\hat{E}^{tot}(z) = 0$ is zero at the boundary z = 0, and at every half wavelength distance away from the boundary in region 1 which is illustrated in Figure (5.2).

Total Magnetic Field Expression:

$$\hat{H}^{tot} = \hat{H}^{i}(z) + \hat{H}^{r}(z) = \left(\frac{\hat{E}_{m1}^{+}}{\hat{\eta}_1} e^{-j\beta_1 z} - \frac{\hat{E}_{m1}^{-}}{\eta_1} e^{j\beta_1 z} \right) a_y$$

The minus sign in the reflected magnetic field expression is simply because for a –z propagating wave the amplitude of the reflected magnetic field is related to that of the reflected electric field by $(-\eta_1)$. Substituting $\hat{E}_{m1}^{-} = -\hat{E}_{m1}^{+}$ yields:

$$\hat{H}^{tot}(z) = \frac{\hat{E}_{m1}^{+}}{\hat{\eta}_1} \left(e^{-j\beta_1 z} + e^{j\beta_1 z} \right) a_y$$

$$= 2\frac{\hat{E}^{+}_{m1}}{\eta_1}\cos\beta_1 z\, a_y \qquad (5.15)$$

The time-domain magnetic field expression is obtained from equation (5.15) as:

$$\hat{H}^{tot}(z,t) = 2\frac{\hat{E}^{+}_{m1}}{\eta_1}\cos\beta_1 z\, a_y \cos\omega t\; a_y \qquad (5.16)$$

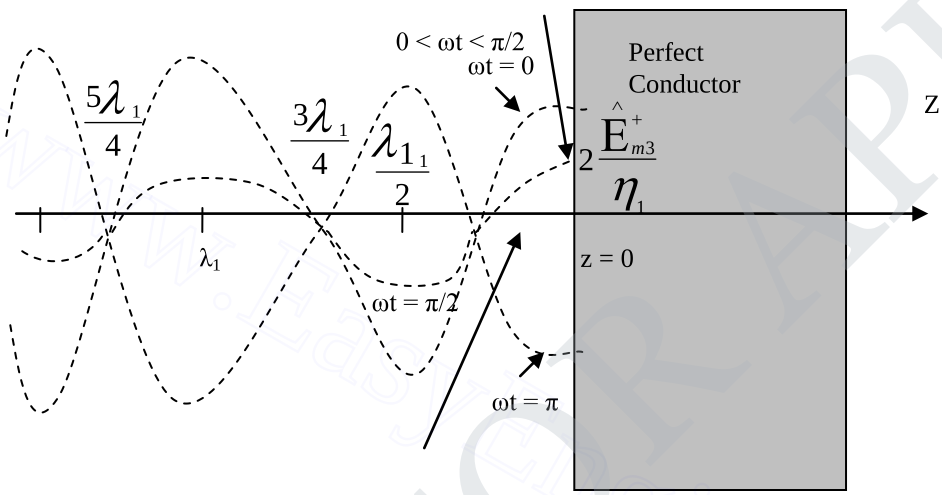


Figure 5.3 The magnetic field distribution in front of a perfect conductor as a function of time.

Equation 5.16 is also a standing wave as illustrated in Figure 5.3, with the maximum amplitude of the magnetic field occurring at the perfect conductor interface (z = 0) where the total electric field is zero. The location of nulls in the magnetic field are where the values of z at $\cos\beta_1 z = 0$, therefore,

$$\beta_1 z = \text{odd number of } \frac{\pi}{2} = (2m+1)\frac{\pi}{2} \quad (m = 0, \pm 1, \pm 2, \ldots \text{ or } z = (2m+1)\frac{\lambda_1}{4}$$

The magnetic field distribution in front of a perfectly conducting boundary is illustrated in Figure 5.3, where we can observe that its first null occurs at $z = \lambda_1/4$. This is the location of the maximum electric field (see Figure 5.2). By comparison, equations (5.13) & (5.16) shows that the electric and magnetic fields of a standing wave are 90° out of time phase due to the sin(ωt) term and cos(ωt), respectively. This will result in a zero average power transmitting in either direction of the standing wave. This can be illustrated by using the complex forms of the fields to calculate the time-average Poynting vector P_{av} (z):

$$P_{ave}(z) = \frac{1}{2}\left[\operatorname{Re}\hat{\mathrm{E}}(z) \wedge \hat{\mathrm{H}}(z)\right]$$
$$= \frac{1}{2}\operatorname{Re}\left[-2j\mathrm{E}_{m1}^{+}\sin\beta_1 z\, a_x \wedge 2\frac{\mathrm{E}_{m1}^{+}}{\eta_1}\cos\beta_1\, z a_y\right]$$
$$= 0 \qquad (5.18)$$

The zero value of P_{av} (z) is obtained because the result of the vector product of is a $\hat{\mathrm{E}}(z) \wedge \hat{\mathrm{H}}^{*}(z)$ is an imaginary number. This zero value of average power transmitted by this wave is yet another reason for calling the total wave in front of the perfect conductor a "standing wave."

Examples:

In free space $\mathrm{E}(z,t) = 50\cos(\omega t - \beta z)a_x \quad (V/m)$. Obtain **H** (z, t).

Examination of the phase, ωt-βz, shows that the direction of propagation is +z, since **E x H** must also be in the +z direction, **H** must have the direction –$\mathbf{a}_x$. Consequently,

$$\frac{\mathrm{E}_y}{\mathrm{H}_x} = \eta_o = 120\pi\,\Omega \qquad \text{Or} \qquad \mathrm{H}_x = \frac{10^3}{120\pi}\sin(\omega t - \beta z) \quad A/m$$

And

$$\mathrm{H}(z,t) = \frac{10^3}{120\pi}\sin(\omega t - \beta z)a_x \quad A/m$$

For the wave of the problem above determine the propagation constant γ, given that the frequency is f = 95.5MHz.

In general, $\gamma = j\omega\mu(\sqrt{\sigma + j\omega\varepsilon})$. In free space, σ = 0, so that

$$\gamma = j\omega\sqrt{\mu_o\,\varepsilon_o} = j\frac{2\pi f}{c} = j\frac{2\pi(95.5*10^6}{3*10^8} = j(2.0)\,m^{-1}$$

This result shows that the attenuation factor is α = 0 and the phase-shift constant is β = 2.0 rad/m.