DEPARTMENT OF MECHANICAL ENGINEERING

MA3251 STATISTICS AND NUMERICAL METHODS *UNIT NOTES*

UNIT- I TESTING OF HYPOTHESIS

DEFINITIONS

Errors in sampling

Type I error: Reject H_0 when H_0 is true

Type II error: Accept H_0 when H_0 is false

Critical Region

The region in the sample space S corresponding to a statistic which Amounts to the rejection of the null hypothesis H_0 is called critical region

Null and Alternative hypothesis

Null hypothesis (H_0) is the hypothesis which is tested for possible rejection under the assumption that it is true. Any hypothesis which is complementary to the null hypothesis is called an alternative hypothesis (H_1)

Type I and Type II errors in testing hypothesis

Type I error : Reject H_0 when H_0 is true

Type II error: Accept H_0 when H_0 is false ie., accept H_0 when H_1 is true

One-Tailed test:

In any test the critical region is represented by a portion of the area under the probability curve of the sampling distribution of the test statistic. A test of any statistical hypothesis where the alternative hypothesis is one tailed (right or left tailed) is called a one tailed test.

Two- Tailed test:

A test of statistical where the alternative hypothesis is two tail.
H_0: $\mu = \mu_0$ against the alternative hypothesis $H_1 : \mu > \mu_0$ *and* $H_1 : \mu < \mu_0$
is known as two tailed test in such case the critical region is given by the portion of the area lie in the both tails of the probability curve of the test statistic

Standard error

The standard deviation of the sampling distribution of a statistic is known as its standard error.

Problems

1. A coin is tossed 256 times and 132 heads are obtained. Would you conclude that the coin is a biased one?

$$p = \frac{1}{2} = 0.5 \; q = 0.5$$

$$p' = \frac{132}{256} = 0.52$$

$$z = \frac{p' - p}{\sqrt{\frac{pq}{n}}}$$

$$\therefore |z| = \left| \frac{0.52 - 0.5}{\sqrt{(0.5)(0.5)/256}} \right| = 0.4992 < 1.96$$

$\therefore$ The hypothesis H_0: the coin is unbiased is accepted at 5% level.

2. A sample of 900 members has a mean 3.4 and S.D 2.61. Is the sample from a large population mean 3.25 and S.D is 2.61/

$$H_0 : \mu = 3.25$$

$$H_1 : \mu \neq 3.25$$

the test statistic,

$$|z| = \left| \frac{\bar{x} - \mu}{s/\sqrt{n}} \right| = \frac{(0.15)(30)}{2.61} = 1.724$$

Since $|z| < 1.96, H_0$ is accepted at 5% level. The sample is drawn from the Population with mean 3.25

3. The mean of two sample large samples of 1000 and 200 members are 67.5 inches and 68 inches respectively. Can the samples be regard as drawn from the population of standard deviation of 2.5 inches? Test at 5% Los

Solution:

(i) We set up $H_0 : \mu_1 = \mu_2$;

ie., the difference is not significant

$H_1 : \mu_1 \neq \mu_2$

Here it is two tailed test

$$\text{Test statistic } Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{582 - 546}{\sqrt{\frac{(24)^2}{100} + \frac{(28)^2}{100}}}$$

$$\therefore \; Z = \frac{360}{\sqrt{(24)^2 + (28)^2}} = 9.76$$

$$\Rightarrow \; |Z| = 9.76$$

Critical value :

At 5% Los, the table value of Z is 1.96

Conclusion:

Since |Z| > 1.96, H_0 is rejected at 5% Los.

∴ The sample cannot be regards as drawn from the same population.

4. What are the uses of t-distribution?

(i) To test if the sample mean $\bar{x}$ differs significantly from the hypothetical value μ of the population mean.

(ii) To test the significance of the difference between two sample means.

(iii) To test the significance of an observed sample correlation coefficient and sample regression coefficient.

5. Given $n_1 = n_2 = 20, \bar{x}_1 = 220, \bar{x}_2 = 250, s_1 = 20, s_2 = 25$. Compute the standard error for $\bar{x}_1 - \bar{x}_2$

$$S.E(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{s_1^2 + s_2^2}{n-1}}$$

$$\sqrt{\frac{400 + 625}{}} \; 7.345$$

6. A random sample of size 16 has 53 as mean and the sum of the squares of the deviation taken from Mean is 150. Can this sample be regarded as taken from the population having 56 as mean? (t for 15 d.f at 5% level=2.13)

the test statistic,

$$t = \frac{\bar{x} - \mu}{S/\sqrt{n}} --------(1)$$

$$where\ S = \sqrt{\frac{1}{n-1}\sum (x-\bar{x})^2} = \sqrt{\frac{150}{15}}$$

$$S = 3.162$$

$$(1) \Rightarrow t = \frac{(53-56)^2}{3.162} = 3.8$$

$$|t| = 3.8$$

Since $|t| < 2.13, H_0$ *is rejected*. $\therefore$ The sample is not drawn from the population Having 56 as mean

7. Two independent samples from normal pop's with equal variances gave the following results

Samp	Size	Mean	S.D
1	16	23.4	2.5
2	12	24.9	2.8

Test for the equations of means.

Solution:

(i) We set up $H_0 : \mu_1 = \mu_2$; ie., there is no significant difference between their means

$H_1 : \mu_1 \neq \mu_2$

Los: $\alpha = 0.05$

Test Statistic $t = \dfrac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\dfrac{1}{n_1}+\dfrac{1}{n_2}}}$ Where $S^2 = \dfrac{n_1 s_1{}^2 + n_2 s_2{}^2}{n_1 + n_2 - 2}$

Given $\overline{x_1} = 23.4;\ n_1 = 16;\ s_1 = 2.5$

$\overline{x_2} = 24.9;\ n_2 = 12;\ s_2 = 2.8$

$$S^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{16(2.5)^2 + 12(2.8)^2}{16 + 12 - 2}$$

$$= \frac{100 + 94.08}{26} = 7.465$$

$S = 2.732$

$$\therefore\ t = \frac{23.4 - 24.9}{2.732\sqrt{\frac{1}{16} + \frac{1}{12}}} = -1.438$$

$\therefore\ |t| = 1.438$

.

Number of degrees of freedom $= n_1 + n_2 - 2 = 26$

Critical value :

The table value of t for 26 d.f at 5% Los is

$t_{0.05} = 2.056$

Conclusion:

Since the calculated value of t is less than table value of t,

H_0 is accepted at 5% Los.

$\therefore$ There is no significant difference between their means

8. State the conditions for applying χ^2-test

(iv) The sample observations should be independent.

(v) The constraints on cell frequencies must be linear

(vi) Total frequency N>30

(vii) No theoretical frequency should be less than 5.

9. What is goodness of fit test?

Chi-Square test enables us to ascertain how well the theoretical distributions such as Binomial, Poisson or Normal etc, fit empirical distributions. If calculated χ^2 < tabulated value, the fit is considered to be good. If the calculated χ^2 >table value, the fit is considered to be poor.

$$\text{The value of } \chi^2 = \frac{\sum (O_i - E_i)^2}{E_i}$$

O_i : *Observed frequency*

E_i : *Expected frequency*

10. Calculate χ^2 for the following data

O_i : 37 44 19

E_i : 31 38 31

$$\chi^2 = \frac{\sum (O_i - E_i)^2}{E_i} = \frac{36}{61} + \frac{36}{38} + \frac{144}{31} = 6.76$$

11. For 2x2 consistency table

a	b
c	d

Write the formula for χ^2

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+c)(b+d)(a+b)(c+d)}$$

12 Find the value of χ^2 for the data in the consistency table.

2	10
8	4

$$\chi^2 = \frac{(8-80)^2(24)}{(10)(14)(12)(12)} = 6.17$$

13. The following table gives the number of aircraft accident that occured during the various days of the week. Test whether the acidents are uniformly distributed over the week.

Days	Mon	Tue	Wed	Thu	Fri	Sat	Total
No.of accidents	14	18	12	11	15	14	84

Solution:

We set up H_0 : The accidents are uniformly distributed over the week

Los $\alpha = 0.05$

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i}{E_i}^2$

Under the nul hypothesis,

The expected frequency of the on each day = $\frac{84}{6} = 14$

O_i : 14 18 12 11 15 14

E_i : 14 14 14 14 14 14

$$\chi^2 = \frac{14-14^{\ 2}}{14} + \frac{18-14^{\ 2}}{14} + \frac{12-14^{\ 2}}{14} + \frac{11-14^{\ 2}}{14} + \frac{15-14^{\ 2}}{14} + \frac{14-14^{\ 2}}{14}$$

$= 1.143 + 0.286 + 0.643 + 0.071$

$= 2.143$

Number of degrees of freedom V = n – 1=7 – 1=6

Critical value:

The tablulated value of χ^2 at 5% for 6 d.f is 12.59

Conclusion:

Since $\chi^2 < 12.59$, we accept the null hypothesis

∴ We conclude that the accidents are uniformly distributed over the week.

14. What do you mean by Level of Significance?

In testing the given hypothesis, the maximum probability with which we Conclude be willing to risk is called Level of Significance of the test

15. Write the applications of F-test and chi-square test.

Application of F-test

(i) To test whether there is any significant difference between two estimates of population variance

(ii) To test if the two sample have come from the same population.

Application of Chi-square test:

(i) To test the significance of discrepancy between experimental values and the theoretical values

16. Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 and 91 respectively. Compute the value of F.

$$S_1^2 = \frac{160}{8} = 20;\ S_2^2 = \frac{91}{7} = 13$$

$here\ S_1^2 > S_2^2,$

$$F = \frac{S_1^2}{S_2^2} = \frac{20}{13} = 1.5385$$

$$F = 1.5385$$

17. It is known that the mean diameters o rivets produced by two firms A and B are practically the same but the standard deviations may differ.

For 22 rivets produced by A, the S.D is 2.9 m, while for 16 rivets manufactured by B, the S.D is 3.8 m. Test whether the products of A have the same variability as those of B

Solution:

$$H_0: \sigma_1^2 = \sigma_2^2$$

ie., variability for the two types of products are same.

Los: $\alpha = 0.05$ (or) 5%

The test statistic $F = \frac{S_1^2}{S_2^2} \quad S_1^2 > S_2^2$

Given, $n_1 = 22; \quad n_2 = 16$

$S_1 = 2.9; \quad S_2 = 3.8$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{22(2.9)^2}{22-1} = 8.81$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{16(3.8)^2}{16-1} = 15.40$$

$$F = \frac{S_2^2}{S_1^2} \quad S_2^2 > S_1^2$$

$$= \frac{15.40}{8.81}$$

$$F = 1.748$$

Number of degrees of freedom are $V_1 = 16 - 1 = 15$

$V_2 = 22 - 1 = 21$

Critical value:

At 5% Los, the table value of F at d.f (15,21) is $F = 2.18$

Conclusion:

$F < 2.18$, H_0 is accepted at 5% Los.

$\therefore$ Variability for two types of products may be same.

UNIT-II
DESIGN OF EXPERIMENTS

Design of experiments:

Design of experiments may be defined as the logical construction of the experiment in which the degree uncertainty with which the inference as drawn may be well defined.

The purpose of experimental design is to obtain maximum information of an experiment with the minimum experimental error, cost and labour.

Basic principles of experimental design:

The following three basic principles are adopted when the experiments are designed.

1. Randomization.
2. Replication.
3. Local control.

Randomization

Randomization is a technique by which two or more treatments are applied to different group of the sample, the treatments to be applied to any group being decided by random sampling technique.

Replication

Replication means the repetition of an experiment. It means the repetition of the treatment under investigation. It helps us in estimating the experimental error. Thus, the repetition of treatments results in more reliable estimate than is possible with single observation.

Local control

The process of reducing the experimental error by dividing the relatively the heterogeneous experimental area into homogeneous is known as local control.

we use local control which involves "*grouping*"(combining homogenous set of plots) , "*blocking*"(assigning same number of plots in different blocks) and "*balancing*"(adjusting grouping and blocking procedure) of experimental units.

ANALYSIS OF VARIANCE

Analysis of variance is technique, which is used to test the equality of three or more population means by comparing the sample variances using F-distribution. This technique split up the variance into its various components, usually into two parts

i. Variance between the samples.
ii. Variance within the samples.

Completely Randomized Design (C.R.D):

Suppose that there are *N* plots available & we wish to compare *h* treatments for an experiment and let the i^{th} treatment be repeated (replicated) n_i times such that $n_1 + n_2 + \ldots + n_h = N$. The *N* plots are selected randomly and each given a treatment to get the design which is completely random. This design is used only if plots used are homogenous.

Analysis of variance (ANOVA) for one way classification (or) one factor experiments:

Procedure:

Let N-Total number of items in the given data.

K- Number of treatment.

n-Number of samples in each treatment.

STEP 1: Set up Null Hypothesis H_0

Set up Alternative Hypothesis H_1

<u>**Prepare Table – I using the following steps:**</u>

STEP 2: Find Sum of all the items of the sample $G = \sum T_1 + \sum T_2 + \cdots + \sum T_K$ where K isnumber of Treatments, T-total and G-Grand total.

STEP 3: FindSum of squares of all the items of eachtreatment $S.S = \sum_1 {X_{ij}}^2 + \sum_2 {X_{ij}}^2 + \cdots + \sum_k {X_{ij}}^2$

STEP 4: Find the Correction Factor (C.F) = $\frac{G^2}{N}$ (G-Grand total, N-Total number of obsevations)

Table – I (calculations for ANOVA):

STEP 5: Find Total sum of squares (T.S.S) = Sum of squares of all the items - Correction Factor

$$= S.S - \frac{G^2}{N} = \sum\sum {X_{ij}}^2 - \frac{G^2}{N}$$

STEP 6: Find Sum of squares between samples (SSB) = $\sum \frac{{T_i}^2}{n_i} - C.F$

$$= \sum \frac{{T_i}^2}{n_i} - \frac{G^2}{N}$$

STEP 7: Find Degrees of freedom $\upsilon_1 = K - 1, \upsilon_2 = N - K$

STEP 8: Find Mean square between samples

(MSB)= $\frac{\text{Sum of squares between samples (SSB)}}{(\text{Degrees of freedom } \upsilon_1 = K-1)}$

STEP9: Find Sum of squares within samples (SSW) = Total sum of squares (T.S.S) - Sum of squares between samples(SSB)

STEP 10: FindMean square within samples

(MSW) = $\frac{\text{Sum of squares within samples (SSB)}}{(\text{Degrees of freedom } \upsilon_2 = N-K)}$

STEP 11:Table-II (ANOVA Table):

Source of variation	Sum of squares	Degree of freedom	Mean squares	F-Ratio
Between samples	SSB	$\upsilon_1 = K - 1$	$MSB = \frac{SSB}{\upsilon_1}$	$F_C = \frac{MSB}{MSW}$
Within samples	SSW	$\upsilon_2 = N - K$	$MSW = \frac{SSW}{\upsilon_2}$	$F_C = \frac{MSW}{MSB}$

Where,

SSB = **Sum of squares between samples**

SSW = **Sum of squares within samples**

MSB = **Mean square between samples**

MSW= **Mean square within samples**

= **Calculated F – value**

$\frac{MSB}{MSW}$ If ;B $> MSW$ (or)

$= \frac{MSW}{MSB}$ If MSW $> MSB$

Type-1:Reframe the table by applying the principles of experimental design.

1. A completely randomized design experiment with 10 plots and 3 experiments gave the following result.

Plot No.	1	2	3	4	5	6	7	8	9	10
Treatments	A	B	C	A	C	C	A	B	A	B
Yield	5	4	3	7	5	1	3	4	1	7

Analyse the result for treatment effects.

Solution:

Let us reframe the table by applying the principles of experimental design.

Given data

Total number of items N = 10. N umber of Treatments K = 3.

n- Number of samples in each treatment (not equal in this problem).

$i.e, n_1 = 4, n_2 = 3, n_3 = 3$

Null Hypothesis H_0: There is no significant difference among the average yields in the 3 treatments.

AlternativeHypothesis H_1: There is significant difference among the average yields in the 3 treatments.

Table – I (calculations for ANOVA):

Total sum of squares (T.S.S) = $= \sum\sum {X_{ij}}^2 - \frac{G^2}{N} = 200 - 160 = 40$

Sum of squares between samples (SSB) $= \sum \frac{{T_i}^2}{n_i} - \frac{G^2}{N}$

$= 166 - 160 = 6$

Sum of squares within samples (SSW) $= T.S.S - SSB = 40 - 6 = 34$

Treatments K=3	Yield of the plots / Number of Blocks 1	2	3	4	Row total T_i	$\frac{T_i^2}{n_i}$	$\sum X_{ij}^2$
A	5	7	3	1	16	$\frac{16^2}{4} = 64$	84
B	4	4	7	-	15	$\frac{15^2}{3} = 75$	81
C	3	5	1	-	9	$\frac{9^2}{3} = 27$	35
	C.F $== \frac{40^2}{10} = 160$				Grand total $T = \sum T_i = 40$	$\sum \frac{T_i^2}{n_i} = 166$	$\sum X_{ij}^2 = 200$

Source of variation	Sum of squares	Degree of freedom	Mean squares	F-Ratio
Between samples	SSB = 6	$v_1 = 3 - 1 = 2$	$MSB = \frac{SSB}{v_1} = \frac{6}{2} = 3$	$F_C = \frac{MSW}{MSB} = \frac{4.86}{3} =$ 1., (MSW > MSB)
Within samples	SSW = 34	$v_2 = 10 - 3 = 7$	$MSW = \frac{SSW}{v_2} = \frac{34}{7} = 4.86$	

∴Calculated value of F is 1.62

∴There is no significant difference among the average yields in the 3 treatments.

Two Way Classifications:

Randomized block design (R.B.D):

ANOVA – Table:

Source of variation	Sum of squares	Degree of freedom	Mean squares	F-Ratio
Column treatments	SSC	$v_1 = c - 1$	$MSC = \frac{SSC}{c-1}$	$F_C = \frac{MSC}{MSE}$
Row treatments	SSR	$v_2 = r - 1$	$MSR = \frac{SSR}{r-1}$	$F_R = \frac{MSR}{MSE}$
Error (or) Residual	SSE	$v_3 = (r-1) \times (c-1)$	$MSE = \frac{SSE}{(r-1)(c-1)}$	

Where,

SSC = **Sum of squares between blocks (columns)**

SSR = **Sum of squares between varieties (rows)**

MSC = **Mean square between blocks (columns)**

MSR = **Mean square between varieties (rows)**

SSE = **Residual Sum of squares = T.S.S – (S.S.C + S.S.R)**

MSE = **Mean square error** $MSE = \frac{SSE}{(r-1)(c-1)}$

1. **The yield of four strains of a particular variety of wheat was planted in five randomized blocks in Kgs per plot is given below.**

Strains	Blocks				
	1	2	3	4	5
A	32	34	34	35	36
B	33	33	36	37	34
C	30	35	35	32	35
D	29	22	30	28	28

Test for difference between blocks and difference between strains.

ANOVA table

Source of variation	Sum of squares	Degree of freedom	Mean squares	F-Ratio
Column treatments	SSC = 27.3	$\upsilon_1 = c-1 = 5-1 = 4$	$MSC = \frac{SSC}{\upsilon_1} = \frac{27.3}{4} = 6.825$	$F_C = \frac{MSC}{MSE} = \frac{6.825}{4.59} = 1.486$
Row treatments	SSR = 170.4	$\upsilon_2 = r-1 = 4-1 = 3$	$MSR = \frac{SSR}{\upsilon_2} = \frac{170.4}{3} = 56.8$	$F_R = \frac{MSR}{MSE} = \frac{56.8}{4.59} = 12.37$
Error (or) Residual	SSE = 55.1	$\upsilon_3 = (r-1)(c-1) = 4 \times 3 = 12$	$MSE = \frac{SSE}{\upsilon_3} = \frac{55.1}{12} = 4.59$	*

Conclusion: (i)For Blocks:

Tabulated value of for (4, 12)at 5% l.o.s is 3.26.

Calculated value of is $= 1.486$

Fc <Null hypothesis H_0 (i) is accepted.

(ii) For Strains:

Tabulated value of for (3, 12)at 5% l.o.s is 3.49.

Calculated value of is $= 12.37$

F_R >Null hypothesis H_0(ii)is rejected.

Latin square design:

Advantages:

1. Latin square arrangement design controls more of the variation than the CRBD with a two way stratification.
2. The analysis is simple.
3. Even with missing data the analysis remains relatively simple.

ANOVA TABLE

Source of variation	Sum of squares	Degree of freedom	Mean squares	F-Ratio
Columns	SSC	$n-1$	$MSC = \frac{SSC}{n-1}$	$F_C = \frac{MSC}{MSE}$
Rows	SSR	$n-1$	$MSR = \frac{SSR}{n-1}$	$F_R = \frac{MSR}{MSE}$
Treatments	SST	$n-1$	$MST = \frac{SST}{n-1}$	$F_T = \frac{MST}{MSE}$
Error (or) Residual	SSE	$(n-1)(n-2)$	$MSE = \frac{SSE}{(n-1)(n-2)}$	*

Where,
SSC = **Sum of squares between columns**
SSR = **Sum of squares between rows**
SST = **Sum of squares between rows**
MSC = **Mean square between treatments**
MSR = **Mean square between rows**
MST = **Mean square between treatments**
SSE = **Residual Sum of squares = T.S.S – (S.S.C + S.S.R+S.S.T)**
MSE = **Mean square error** $MSE = \frac{SSE}{(n-1)(n-2)}$

Problems:

1. **Analyze the following results of a Latin square experiments:**

	1	2	3	4
1	A(12)	D(20)	C(16)	B(10)
2	D(18)	A(14)	B(11)	C(14)
3	B(12)	C(15)	D(19)	A(13)
4	C(16)	B(11)	A(15)	D(20)

The letter A, B, C, D denotes the treatments and the figures in brackets denote the observations

Solution:

Total number of items in the given data N = 16.Number of columns c= 4.Number of rows r = 4.

Null Hypothesis H_0:

(i) There is no significant difference between**columns.**

(ii)There is no significant difference between**rows** and

(iii) There is no significant difference between**treatments.**

AlternativeHypothesis H_1: :

(i) There is significant difference between**columns.**

(ii)There is significant difference between**rows** and

(iii) There is significant difference between**treatments**

Source of variation	Sum of squares	Degree of freedom	Mean squares	F-Ratio
Columns	SSC = 2.5	$n-1=3$	$MSC=\frac{SSC}{n-1}=\frac{2.5}{3}=0.87$	$F_C=\frac{MSE}{MSC}=\frac{1.08}{0.87}=1.24$
Rows	SSR = 3.5	$n-1=3$	$MSR=\frac{SSR}{n-1}=\frac{3.5}{3}=1.167$	$F_R=\frac{MSR}{MSE}=\frac{1.167}{1.08}=1.081$
Treatments	SST = 144.5	$n-1=3$	$MST=\frac{SST}{n-1}=\frac{144.5}{3}=48.17$	$F_T=\frac{MST}{MSE}=\frac{48.17}{1.08}=44.60$
Error (or) Residual	SSE = 6.5	$(n-1)(n-2)=(4-1)(4-2)=3\times 2=6$	$MSE=\frac{SSE}{(n-1)(n-2)}=\frac{6.5}{6}=1.08$	*

Conclusion:(i)For Columns: Tabulated value of$F$ for (6, 3) at 5% l.o.s is 8.94. Calculated value of F is $F_C=1.24$.$\because F_C < F_T$ Null hypothesis H_0 (i)is accepted.

(ii)For Rows: Tabulated value of$F$ for (3, 6) at 5% l.o.s is 4.76. Calculated value of F is $F_R=1.081 \because F_R < F_T$ Null hypothesis H_0 (ii)is accepted.

(iii)For Treatments: Tabulated value of$F$ for (3, 6) at 5% l.o.s is 4.76. Calculated value of F is $F_T=44.60 \because F_T > F_T$ Null hypothesis H_0 (iii)is rejected.

2^2-Factorial Design:

In the 2^2-Factorial Design we will have 2 factors each at two levels (0, 1) (say). Then there are 2×2 i.e., 4 treatment combinations in all. *This 2^2-Factorial Design can be performed in the form of* **CRD, RBD, LSD**.By Yate's notation "**K**" and "**P**" denote the 2 factors under study and 'k' and 'p' denote one of the 2 levels of each corresponding factors and this is called second level. The first level of "**K**" and "**P**" is generally expressed by the absence of the corresponding letter in the treatment combinations.

The four treatment combinations are,

$\boldsymbol{k_0p_0(or)1}$ – Factors **"K"** and **"P"** both at first level.

$\boldsymbol{k_0p_0(or)1}$ – **"K"** at second level and **"P"**at first level.

$\boldsymbol{k_0p_0(or)1}$ – **"K"** at first leveland **"P"**at second level

$\boldsymbol{k_0p_0(or)1}$ – Factors **"K"** and **"P"** both at second level.

NOTE:

1. $]=[kp]-[p]+[k]-[1]$
2. $]=[kp]+[p]-[k]-[1]$
3. $\boldsymbol{P}]=[kp]-[p]-[k]+[1]$

1. An experiment was planned to study the effect of sulphate of potash & super phosphate on the yield of potatoes. All the combinations of 2 levels of super phosphate (0 cent (p_0) & 0 cent (p_1) / acre) were studied of potash (0 cent (k_0) and 0 cent (k_1) / acre) were studied in a randomized block design with 4 replications for each.The (1 / 70) obtained are given in the following table. Analyze the data & give your conclusions at 1% level

Block	Yields (l_b per plot)			
I	(1)	*k*	*p*	*kp*
	23	25	22	38
II	*p*	(1)	*k*	*kp*
	40	26	36	38
III	(1)	*k*	*kp*	*p*
	29	20	30	20
IV	*kp*	*k*	*p*	(1)
	34	31	24	28

Table-II (ANOVA table):

Source of variation	Degree of freedom	Sum of squares	Mean squares	F-Ratio
k	1	$S_K = 100$	$MS_K = \frac{S_K}{D.f} = 100$	$F_K = \frac{MS_K}{MSE} = \frac{100}{25.5} = 3.92$
p	1	$S_p = 49$	$MS_p = \frac{S_p}{D.f} = 49$	$F_P = \frac{MS_p}{MSE} = \frac{49}{25.5} = 1.92$
kp	1	$S_{KP} = 49$	$MS_{KP} = \frac{S_{KP}}{D.f} = 49$	$F_{KP} = \frac{MS_{KP}}{MSE} = \frac{49}{25.5} = 1.92$
Error	$N - c - r + 1 = 16 - 4 - 4 + 1 = 9$	SSE = 229.5	$MSE = \frac{SSE}{D.f} = \frac{229.5}{9} = 25.5$	*

Conclusion:

F_{Tab} for (1, 9) at 1% l.o.s is 10.56.
Since < in all three cases H_0 is accepted. ie., There is no significant difference between blocks and treatments.

UNIT – III SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

NEWTON'S METHOD

The Newton-Raphson method in one variable is implemented as follows:

Given a function f defined over the reals x, and its derivative $f\,'$, we begin with a first guess x_0 for a root of the function f. Provided the function is reasonably well-behaved a better approximation x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}\,.$$

Geometrically, $(x_1, 0)$ is the intersection with the x-axis of a line tangent to f at $(x_0, f(x_0))$.

The process is repeated as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

until a sufficiently accurate value is reached.

ITERATIVE FORMULA:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{Where n=0, 1, 2, 3............}$$

PROBLEMS RELATED TO NEWTON-RAPHSON METHOD

1. Apply Newton's method to find the positive root of $x^2 = 6x - 4$ to two decimal places.

SOLUTION:

STEP1: Given, $x^2 = 6x - 4$.

Let f(x) = $x^2 - 6x + 4$ & $f'(x) = 2x - 6$

STEP2: To find the positive root of $x^2 = 6x - 4$.

STEP3: Intervel of roots:

f(x) = $x^2 - 6x + 4$

f(0) = 4

f(1) = -1

Therefore , the root lies between 0 and 1.

Let x_0=1

STEP4: **1st iter eation**: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Here, x_0=1, $f(x_0) = x_0^2 - 6x_0 + 4$ = -1, $f'(x_0) = 2x_0 - 6$ =-3

x_1 = 0.67

2nd itereation: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Here, x_1 = **0.67**, $f(x_1) = x_1^2 - 6x_1 + 4$ = .28, $f'(x_1) = 2x_1 - 6$ =-4.65

x_2 = 0.73

3rd itereation: $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

Here, $x_2 = 0.73$, $f(x_2) = x_2^{\,2} - 6x_2 + 4 = .009$, $f'(x_2) = 2x_2 - 6 = -4.4013$

$x_1 = 0.73$

Therfore, $x_1 = x_2 = 0.73$

RESULT: x=0.73

Show that the Newton Raphson formula to find $\sqrt{a}$ can be expressed in the form $x_{n+1} = \frac{1}{2}\left[x_n + \frac{a}{x_n}\right]$,n=0,1,2,3…..

Sol:

If $x = \sqrt{a}$

$x^2 = a$

$x^2 - a = 0$

$f(x) = x^2 - a$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^{\,2} - a}{2x_n} = \frac{x_n^{\,2} + a}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{a}{x_n}\right], \text{ n} = 0, 1, 2, 3\ldots..$$

3. Show that the iterative formula for finding the reciprocal of N is $x_{n+1} = x_n(2 - N x_n)$.

Sol:

Let $x = \frac{1}{N}$

$N = \frac{1}{x}$

$f(x) = \frac{1}{x} - N$

$f'(x) = -\frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^{\,2}}}$$

$$= x_n + x_n^{\,2}\left(\frac{1}{x_n} - N\right)$$

$$= 2x_n - N x_n^{2}$$

$$x_{n+1} = x_n (2 - N x_n).$$

GAUSSIAN ELIMINATION

PROBLEMS RELATED TO GAUSS ELIMINATION METHOD

1.Apply Gauss elimination method to solve the equation 10x-2y+3z=23,2x+10y-5z=-33,3x-4y+4z=41.

SOLUTION:

STEP1: **Given,** 10x-2y+3z=23,
2x+10y-5z=-33,
3x-4y+4z=41.

STEP2: **To solve** 10x-2y+3z=23, 2x+10y-5z=-33, 3x-4y+4z=41 using Gauss elimination method.

STEP3:

$$[A,I] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$$

$$\approx \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 3 & -4 & 10 & 41 \end{bmatrix} \quad R_2\text{-}5R_2\text{-}R_1$$

$$\approx \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix} \quad R_3\text{-}10R_3\text{-}3R_1$$

$$\approx \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_3\text{-}52R_3\text{+}34R_2$$

This is an upper triangular matrix.
Therfore, 3780z = 11340
z = 3,
52y-28z=-188,
y =-2.
10x-27+3z=23,
x =1

RESULT: **x =1 y =-2. z = 3,**

APPLICATION TO FINDING INVERSES:

If Gauss–Jordan elimination is applied on a square matrix, it can be used to calculate the matrix's inverse. This can be done by augmentingthe square matrix with the identity matrix of the same dimensions and applying the following matrix operations:

$$[AI] \Rightarrow A^{-1}[AI] \Rightarrow [I A^{-1}].$$

If the original square matrix, *A*, is given by the following expression:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Then, after augmenting by the identity, the following is obtained:

$$[AI] = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

By performing elementary row operations on the [*AI*] matrix until it reaches reduced row echelon form, the following is the final result:

$$[IA^{-1}] = \begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

The matrix augmentation can now be undone, which gives the following:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

A matrix is non-singular (meaning that it has an inverse matrix) if and only if the identity matrix can be obtained using only elementary row operations.

GAUSS –JORDAN METHOD

WORKING PROCEDURE:

Consider, $a_1x+b_1y+c_1z=d_1$,

$a_2x+b_2y+c_2z=d_2$,

$a_3x+b_3y+c_3z=d_3$,

It is equivalent to AX=B where $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $B=\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Step: 1 To write the augmented matrix [A,I]

Step: 2 To reduce the augmented matrix [A,I] into a diagonal matrix using the elementary row operations

Step: 3 To solve the matrix without using back substitution method

PROBLEMS RELATED TO GAUSS JORDAN METHOD

1.Solve the system of equations10x-2y+3z=23, 2x+10y-5z=-33, 3x-4y+4z=41.By Gauss Jordan method.

SOLUTION:

STEP1: **Given,** 10x-2y+3z=23,

3x-4y+4z=41.

STEP2: **To solve** 10x-2y+3z=23, 2x+10y-5z=-33, 3x-4y+4z=41 using Gauss Jordan method.

STEP3: $[A,I] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$

$\approx \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 3 & -4 & 10 & 41 \end{bmatrix}$ R_2-$5R_2$-R_1

$\approx \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix}$ R_3-$10R_3$-$3R_1$

$\approx \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$ R_3-$52R_3$+$34R_2$

$\approx \begin{bmatrix} 12600 & -2520 & 0 & 17640 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$ R_1-$1260R_1$-R_3

$\approx \begin{bmatrix} 12600 & -2520 & 0 & 17640 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$ R_2-$135R_2$+R_3

$\approx \begin{bmatrix} 88452000 & 0 & 0 & 88452000 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$ R_1-$7020R_1$+$2520R_2$

This is a diagonal matrix.

z = 3,

y =-2.

x = 1

RESULT: **x =1 y =-2. z = 3,**

2. Solve the principle used in Gauss Jordan method 5x+4y = 15, 3x+7y = 12.

Sol:

The given system is equivalent to $\begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$

A X = B

$[A, B] = \begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 15 \\ 12 \end{bmatrix}$

$[A, B] \sim \begin{bmatrix} 5 & 4 \\ 0 & 23 \end{bmatrix}\begin{pmatrix} 15 \\ 15 \end{pmatrix}$ $R_2 -> 5R_2 - 3R_1$

$\sim \begin{bmatrix} 115 & 0 \\ & \end{bmatrix}\begin{pmatrix} 285 \\ 15 \end{pmatrix}$ $R_1 -> 23R_1 - 4R_2$

115 x = 285
x = 2.4783
23 y = 15
y = 0.6522.

JACOBI METHOD
DESCRIPTION:

Given a square system of *n* linear equations: $A\mathbf{x} = \mathbf{b}$

where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then *A* can be decomposed into a diagonal component *D*, and the remainder *R*:

$$A = D + R \qquad \text{where} \qquad D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix},$$

The element-based formula is thus:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right), \quad i = 1, 2, \ldots, n.$$

Note that the computation of $x_i^{(k+1)}$ requires each element in $\mathbf{x}^{(k)}$ except itself. Unlike the Gauss–Seidel method, we can't overwrite $x_i^{(k)}$ with $x_i^{(k+1)}$, as that value will be needed by the rest of the computation. The minimum amount of storage is two vectors of size *n*.

WORKING PROCEDURE:

Consider the system of equations

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (I)$$

(Check the dominance condition)This system of equations can also be written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

the init $^{(0)}$ $^{(0)}$ $^{(0)}$ ial values of x, y, z respectively.

$$\left.\begin{aligned} x^{(1)} &= \frac{1}{a_1}\left(d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right) \\ y^{(1)} &= \frac{1}{b_2}\left(d_2 - a_2 x^{(0)} - c_2 z^{(0)}\right) \\ z^{(1)} &= \frac{1}{c_3}\left(d_3 - a_3 x^{(0)} - b_3 y^{(0)}\right) \end{aligned}\right\}\text{.........................(3)}$$

Again using these values $x^{(1)}, y^{(1)}, z^{(1)}$ in equation (2)

This process may be repeated till the difference between two consecutive approximations is negligible.

PROBLEMS RELATED TO GAUSS JACOBI METHOD

1. Solve by Jacobi iteration method correct to two decimal places 27x+6y-z=85,x+y+54z=110,6x+15y+2z=72.

SOLUTION:

STEP1:Given,

27x+6y-z=85,
x+y+54z=110,
6x+15y+2z=72.
} (1)

This is not a diagonally dominant.

So, rearrange the given equations,

27x+6y-z=85,.........................(1)

6x+15y+2z=72, ...(2)

x+y+54z=110, ...(3)

This is a diagonally dominant.

From (1), x=(85-6y+z)/27,

From (2), y=(72-6x-2z)/15

From (3), z=(110-x-y)/54.

Let, x=y=z=0

STEP2: (1st iteration)

$X^{(1)}$ =(85-6y+z)/27=85/27=3.148.

$Y^{(1)}$= (72-6x-2z)/15=72/15=4.8.

$Z^{(1)}$=(110-x-y)/54=110/54=2.037.

STEP3: (2nd iteration)

$X^{(2)}$ =(85-6$y^{(1)}$ +$z^{(1)}$)/27=2.157.

$Y^{(2)}$= (72-6$x^{(1)}$ -2$z^{(1)}$)/15=3.269

$Z^{(2)}$=(110-$x^{(1)}$ -$y^{(1)}$)/54=1.890

rd

$X^{(3)} = (85-6y^{(2)} + z^{(2)})/27 = 2.492$

$Y^{(3)} = (72-6x^{(2)} - 2z^{(2)})/15 = 3.685$

$Z^{(3)} = (110-x^{(2)} - y^{(2)}$ $54 = 1.9$

STEP5: **(4th iteration)**

$X^{(4)} = (85-6y^{(3)}+z^{(3)})/27 = 2.401.$

$Y^{(4)} = (72-6x^{(3)}-2z^{(3)})/15 = 3.545$

$Z^{(4)} = (110-x^{(3)}-y^{(3)})/54 = 1.923$

STEP6: **(5th iteration)**

$X^{(5)} = (85-6y^{(4)}+z^{(4)})/27 = 2.432$

$Y^{(5)} = (72-6x^{(4)}-2z^{(4)})/15 = 3.583$

$Z^{(5)} = (110-x^{(5)}-y^{(5)})/54 = 1.927$

STEP7: **(6th iteration)**

$X^{(6)} = (85-6y^{(5)} + z^{(5)})/27 = 2.425$

$Y^{(6)} = (72-6x^{(5)} - 2z^{(5)})/15 = 3.570$

$Z^{(6)} = (110-x^{(5)} - y^{(5)})/54 = 1.926$

Here, $x^5 = x^6 = 2.43$,(correct to 2 decimal places)

$Y^5 = y^6 = 3.57$,(correct to 2 decimal places),

$Z^5 = z^6 = 1.93$,(correct to 2 decimal places).

RESULT: x=2.43,y=3.57,z=1.93

2. Solve the system of equation by
2x-3y+20z = 25,20x+ y - 2z =17, 3x+20y –z = -18 by Gauss Jacobian method
[Only two iteration].

Sol:

As the coefficient matrix is not diagonally domain as it is we rewrite the equation.

20x + y - 2z = 17

3x + 20y – z = -18

2x - 3y + 20 z = 25

Now the diagonal element is domain in the coefficient matrix,

We write x, y, z as following:

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 2x + 3y]$$

Let the initial condition be x = 0, y = 0, z = 0.

First iteration:

$$x_1 = \frac{1}{20.}[17 - y_0 + 2z_0] = 0.85$$

$$y_1 = \frac{1}{20}\left[-18 - 3x_0 + z_0\right] = \frac{1}{20}(-18) = -0.9$$

$$z_1 = \frac{1}{20}\left[25 - 2x_0 + 3y_0\right] = \frac{1}{20}(25) = 1.25$$

Second Iteration:

$x_2 = 1.02$
$y_2 = -0.965$
$z_2 = 1.03.$

GAUSS–SEIDEL METHOD
DESCRIPTION:

Given a square system of *n* linear equations with unknown **x**: $A\mathbf{x} = \mathbf{b}$

where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then *A* can be decomposed into a lower triangular component L_*, and a strictly upper triangular component *U*:

$$A = L_* + U \qquad \text{where} \qquad L_* = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The system of linear equations may be rewritten as:

$$L_*\mathbf{x} = \mathbf{b} - U\mathbf{x}$$

The Gauss–Seidel method is an iterative technique that solves the left hand side of this expression for **x**, using previous value for **x** on the right hand side. Analytically, this may be written as:

$$\mathbf{x}^{(k+1)} = L_*^{-1}(\mathbf{b} - U\mathbf{x}^{(k)}).$$

However, by taking advantage of the triangular form of L_*, the elements of $\mathbf{x}^{(k+1)}$ can be computed sequentially using forward substitution:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j>i} a_{ij}x_j^{(k)} - \sum_{j<i} a_{ij}x_j^{(k+1)}\right), \quad i = 1, 2, \ldots, n.$$

Note that the sum inside this computation of $x_i^{(k+1)}$ requires each element in $\mathbf{x}^{(k)}$ except $x^{(k)}$ itself.

The procedure is generally continued until the changes made by an iteration are below some tolerance.

WORKING PROCEDURE:

This is only refinement of Gauss – Jacobi Method. As before

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$
$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$
$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y).$$

We start with the initial values $y^{(0)}, z^{(0)} = 0$ and get $x^{(1)}, y^{(1)}, z^{(1)}$

First iteration

$$x^{(1)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(0)} - c_1 z^{(0)}\right)$$

$$y^{(1)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(1)} - c_2 z^{(0)}\right)$$

$$z^{(1)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right)$$

Second iteration

$$x^{(2)} = \frac{1}{a_1}\left(d_1 - b_1 y^{(1)} - c_1 z^{(1)}\right)$$
$$y^{(2)} = \frac{1}{b_2}\left(d_2 - a_2 x^{(2)} - c_2 z^{(1)}\right)$$

$$z^{(2)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(2)} - b_3 y^{(2)}\right)$$

The above process may continue for third, fourth etc..., iterations.

PROBLEMS RELATED TO GAUSS- SEIDEL METHOD

1.Solve by **Gauss- Seidel Method** ,correct to two decimal places 27x+6y-z=85, x+y+54z=110, 6x+15y+2z=72.

SOLUTION:

STEP1:Given, 27x+6y-z=85,
x+y+54z=110, (1)
6x+15y+2z=72.

This is not a diagonally dominant.

27x+6y-z=85,……………………….(1)

6x+15y+2z=72, ……………………(2)

x+y+54z=110, ……………………..(3)

This is a diagonally dominant.

From (1), **x=(85-6y+z)/27,**

From (2), **y=(72-6x-2z)/15**

From (3), **z=(110-x-y)/54.**

Lety=z=0

STEP2: **(1st iteration)**

$X^{(1)}=(85-6y^{(0)}+z^{(0)})/27=85/27=3.148.$

$Y^{(1)}=(72-6x^{(1)}-2z^{(0)})/15=72/15=3.541.$

$Z^{(1)}=(110-x^{(1)}-y^{(1)})/54=110/54=1.913$

STEP3: **(2nd iteration)**

$X^{(2)}=(85-6y^{(1)}+z^{(1)})/27=2.157.$

$Y^{(2)}=(72-6x^{(2)}-2z^{(1)})/15=3.269$

$Z^{(2)}=(110-x^{(2)}-y^{(2)})/54=1.890$

STEP4: **(3rd iteration)**

$X^{(3)}=(85-6y^{(2)}+z^{(2)})/27=2.432$

$Y^{(3)}=(72-6x^{(3)}-2z^{(2)})/15=3.572$

$Z^{(3)}=(110-x^{(3)}-y^{(3)}\ 54=1.926$

STEP5: **(4th iteration)**

$X^{(4)}=(85-6y^{(3)}+z^{(3)})/27=2.401.$

$Y^{(4)}=(72-6x^{(4)}-2z^{(3)})/15=3.545$

$Z^{(4)}=(110-x^{(4)}-y^{(4)})/54=1.923$

STEP6: **(5th iteration)**

$X^{(5)}=(85-6y^{(4)}+z^{(4)})/27=2.426$

$Y^{(5)}=(72-6x^{(5)}-2z^{(4)})/15=3.573$

$Z^{(5)}=(110-x^{(5)}-y^{(5)})/54=1.926$

STEP7: **(6th iteration)**

$X^{(6)}=(85-6y^{(5)}+z^{(5)})/27=2.426$

$Y^{(6)}=(72-6x^{(6)}-2z^{(5)})/15=3.573$

$Z^{(6)}=(110-x^{(6)}-y^{(6)})/54=1.926$

Here, $x^5=x^6=2.43$,(correct to 2 decimal places)

$Y^5=y^6=3.57$,(correct to 2 decimal places),

$Z^5=z^6=1.93$,(correct to 2 decimal places).

RESULT: x=2.43,y=3.57,z=1.93

2. **Solve by Gauss Seidel method x-2y =-3, 2x+25y = 15 correct to four decimal places.**

Sol:

$$x-2y=-3$$

$$2x+25y=15$$

$$x=-3+2y$$

$$y=\frac{1}{25}(15-2x)$$

Let be the initial value $y_0=0$

First iteration:

y = - 3 + 2 (0) = -3

$y_1 = \frac{1}{25}[15 - 2x_1] = 0.84$

2^{nd} iteration:

$x_2 = -1.32$

$y_2 = 0.7056$

3 iteration:

$x_3 = -1.5888$

$y_3 = 0.7271$

4 iteration:

$x_4 = -1.5458$

$y_4 = 0.7237$

5 iteration:

$x_5 = -1.5526$

$y_5 = 0.7242$

6 iteration:

$x_6 = -1.5516$

$y_6 = 0.7241$

7 iteration:

$x_7 = -1.5518$

$y_7 = 0.7241$

8 iteration:

$x_8 = -1.5518$

$y_8 = 0.7241$

Hence x = -1.5518 and y = 0.7241

INVERSE OF A MATRIX BY GAUSS JORDAN METHOD

WORKING PROCEDURE:

Consider, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$.

Step: 1 To write the augmented matrix [A,I]

Step: 2 To reduce the augmented matrix [A,I] into [A,X] using the elementary row operations

Then , $X=A^{-1}$

PROBLEMS RELATED TO INVERSE OF THE MATRIX BY GAUSS-JORDAN METHOD

1. Find the inverse of $A=\begin{pmatrix}2&2&3\\2&1&1\\1&3&5\end{pmatrix}$ by Gauss-Jordan method.

SOLUTION:

STEP1: **Given,** $A=\begin{pmatrix}2&2&3\\2&1&1\\1&3&5\end{pmatrix}$

STEP2: **To** Find the inverse of $A=\begin{pmatrix}2&2&3\\2&1&1\\1&3&5\end{pmatrix}$ by Gauss-Jordan method.

STEP3: **[A,I]** $=\begin{bmatrix}2&2&3&1&0&0\\2&1&1&0&1&0\\1&3&5&0&0&1\end{bmatrix}$

$$\approx\begin{bmatrix}1&1&3/2&1/2&0&0\\2&1&1&0&1&0\\1&3&5&0&0&1\end{bmatrix}$$ **R1=R1/2**

$$\approx\begin{bmatrix}1&1&3/2&1/2&0&\\0&-1&-2&-1&1&0\\0&2&7/2&-1/2&0&1\end{bmatrix}$$

$$\approx\begin{bmatrix}1&1&3/2&1/2&0&\\0&1&2&1&-1&0\\0&2&7/2&-1/2&0&1\end{bmatrix}$$

$$\approx\begin{bmatrix}1&0&1/2&-1/2&1&0\\0&1&2&1&-1&0\\0&0&-1/2&-5/2&2&1\end{bmatrix}$$

$$\approx\begin{bmatrix}1&0&1/2&-1/2&1&0\\0&1&2&1&-1&0\\0&0&1&5&-4&-2\end{bmatrix}$$

$$\approx\begin{bmatrix}1&0&0&2&-1&-1\\0&1&0&-9&7&4\\0&0&1&5&-4&-2\end{bmatrix}$$

Hence, $A^{-1}=\begin{bmatrix}2&-1&-1\\-9&7&4\\5&-4&-2\end{bmatrix}$

RESULT: $A^{-1}=\begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$

EIGEN VALUE – POWER METHOD

WORKING PROCEDURE:

Let the initial arbitrary vector of the matrix A whose order is 3 by 3

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{(or)} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{(or)} \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

STEP1: **To write the Given matrix A**

STEP2: **To find $AX_1=\lambda_1 X_2$**

STEP3: **$A X_2=\lambda_1 X_3$**

STEP4: **$A X_3=\lambda_1 X_3$**

STEP5: **$A X_4=\lambda_1 X_5$**

………………………………………………………………

……………………………………………………………..

………………………………………………………………

The above process may continue for third, fourth etc..., upto getting required values.

PROBLEMS RELATED TO POWER METHOD

1. Using Power method find the largest Eigen value and its corresponding Eigen vector $\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}$.

SOLUTION:

STEP1: **Given, A=** $\begin{pmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{pmatrix}$

Let, X_1= $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

STEP2: **AX_1=** $\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$ **=6** $\begin{bmatrix} 0.167 \\ .667 \\ 1 \end{bmatrix}$ **=6 X_2**

STEP3: $A X_2 = \begin{bmatrix} 0.166 \\ 2.336 \\ 8.003 \end{bmatrix} = 8.003 \begin{bmatrix} .021 \\ .292 \\ 1 \end{bmatrix} = 8.003 X_3$

STEP4: $A X_3 = \begin{bmatrix} 1.145 \\ .252 \\ 6.002 \end{bmatrix} = 6.002 \begin{bmatrix} .191 \\ .042 \\ 1 \end{bmatrix} = 6.002 X_4$

STEP5: $A X_4 = \begin{bmatrix} 2.065 \\ -.068 \\ 6.272 \end{bmatrix} = 6.272 \begin{bmatrix} .329 \\ -.011 \\ 1 \end{bmatrix} = 6.272 X_5$

STEP6: $A X_5 = \begin{bmatrix} 2.362 \\ .272 \\ 6.941 \end{bmatrix} = 6.941 \begin{bmatrix} .34 \\ .039 \\ 1 \end{bmatrix} = 6.941 X_6$

STEP7: $A X_6 = \begin{bmatrix} 2.223 \\ .516 \\ 7.157 \end{bmatrix} = 7.157 \begin{bmatrix} .311 \\ .072 \\ 1 \end{bmatrix} = 7.157 X_7$

STEP8: $A X_7 = \begin{bmatrix} 2.095 \\ .532 \\ 7.082 \end{bmatrix} = 7.082 \begin{bmatrix} .296 \\ .075 \\ 1 \end{bmatrix} = 7.082 X_8$

STEP9: $A X_8 = \begin{bmatrix} 2.071 \\ .484 \\ 7.001 \end{bmatrix} = 7.001 \begin{bmatrix} .296 \\ .069 \\ 1 \end{bmatrix} = 7.001 X_9$

STEP10: $A X_9 = \begin{bmatrix} 2.089 \\ .46 \\ 6.983 \end{bmatrix} = 6.983 \begin{bmatrix} .296 \\ .066 \\ 1 \end{bmatrix} = 6.983 X_{10}$

STEP11: $A X_{10} = \begin{bmatrix} 2.1.1 \\ .46 \\ 6.992 \end{bmatrix} = 6.992 \begin{bmatrix} .3 \\ .066 \\ 1 \end{bmatrix} = 6.992 X_{11}$

STEP12: $A X_{11} = \begin{bmatrix} 2.101 \\ .464 \\ 6.998 \end{bmatrix} = 6.998 \begin{bmatrix} .3 \\ .066 \\ 1 \end{bmatrix} = 6.998 X_{12}$

STEP12: $A X_{12} = \begin{bmatrix} 2.102 \\ .464 \\ 6.998 \end{bmatrix} = 6.998 \begin{bmatrix} .3 \\ .066 \\ 1 \end{bmatrix}$

Here, $A X_{11} = A X_{12} = 6.998 \begin{bmatrix} .3 \\ .066 \\ 1 \end{bmatrix}$

UNIT IV

INTERPOLATION, NUMERICAL DIFFERENTIATION AND NUMERICAL INTEGRATION

PROBLEMS BASED ON LAGRANGE'S INTERPOLATION FORMULA

1.Use Lagreange's formula to find the quadratic polynomial that takes these values.

x :	0	1	3
y :	0	1	0

SOLUTION:

Step 1: Given, $x_0 = 0,\ x_1 = 1,\ x_2 = 3$

$y_0 = 0,\ y_1 = 1,\ y_2 = 0$

Step 2: To find find the quadratic polynomial Using Lagreange's formula

Step 3: Required formula,

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 \longrightarrow \textbf{(1)}$$

Step 4: Calculations,

Substitute the values of $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $y_0 = 0$, $y_1 = 1$, $y_2 = 0$ in eqn. (1)

$$\mathbf{Y} = \frac{(x-1)\ (x-3)}{(0-1)(0-3)}0 + \frac{(x-0)\ (x-3)}{(1-0)(1-3)}1 + \frac{(x-0)(x-1)}{(3-0)(3-1)}0$$

$$= \frac{-x^2}{2} + \frac{3x}{2}$$

RESULT : $\mathbf{y} = \frac{-x^2}{2} + \frac{3x}{2}$

2. Find the polynomial f(x) by using Lagrange formula and hence find f(3) for

x	x_0 0	x_1 1	x_2 2	x_3 5
f(x)	2 y_0	3 y_1	12 y_2	147 y_3

Solution:

By Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) +$$

$$\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$y = f(3) = \frac{(3-1)(3-2)(3-5)}{-10}(2) + \frac{(3-0)(3-2)(3-5)}{4}(3) +$$

$$\frac{(3-0)(3-1)(3-5)}{-6}(12) + \frac{(3-0)(3-1)(3-2)}{60}(147)$$

$$= \frac{4}{10}(2) - \frac{6}{4}(3) + 2(12) + \frac{147}{10}$$

$$= 35$$

3. Find the missing term in the following table using Lagrange 's interpolation

x	0	1	2	3	4
y	1	3	9	-	81

Sol:

By Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Let $x_0 = 0$, $y_0 = 1$

$x_1 = 1$, $y_1 = 3$

$x_2 = 2$, $y_2 = 9$

$x_3 = 4$ $y_3 = 81$

$$y = f(x) = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(x-0)(x-1)(x-4)}{(1-0)(1-2)(1-4)}(3) + \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(9) +$$

$$\frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(81)$$

$$y = f(3) = \frac{(3-1)(3-2)(3-4)}{-8}(1) + \frac{(3-0)(3-2)(3-4)}{3}(3) +$$

$$\frac{(3-0)(3-1)(3-4)}{-4}(9) + \frac{(3-0)(3-1)(3-2)}{24}(81)$$

$$= \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4}$$

$$= 31$$

4.Using Lagrange's interpolation formula find y(10) given that y(5) = 12, y(16) = 13,y(9) = 14, y(11) = 16.

Sol:

By Lagrange's interpolation formula

y = f(x) =

$$\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13) +$$

$$\frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) +$$

$$\frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

Putting x = 10

$$y = f(10) = \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-2)}(13) + \frac{(5)(4)(-1)}{(4)(3)(-2)}(14) + \frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

$$= 14.666666$$

4. Obtain the root of f(x) = 0 by Lagrange Inverse interpolation given that f(30) = -30, f(34) = -13,f(42) = 18.

Sol:

By Lagrange's inverse interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1 +$$

$$\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 +$$

$$\frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$

$$=\frac{(y+13)(y-3)(y-18)}{(-30+13)(-30-3)(-30-18)}(30)+\frac{(y+30)(y-3)(y-18)}{(-13+30)(-13-3)(-13-18)}(34)+$$
$$\frac{(y+30)(y+13)(y-18)}{(3+30)(3+13)(3-18)}(38)+\frac{(y+30)(y+13)(y-3)}{(18+30)(18+13)(18-3)}(42)$$

$$=\frac{(y+13)(y^2-21y+54)}{-26928}(30)+\frac{(y+30)(y^2-21y+54)}{8432}(34)+$$
$$\frac{(y-18)(y^2+43y+390)}{-7920}(38)$$
$$+\frac{(y-3)(y^2+43y+390)}{22320}(42)$$

at f(x) = 0 ,y = 0 we get

$$=-\frac{21060}{26928}+\frac{55080}{8432}+\frac{266760}{7920}-\frac{49140}{22320}$$

$= 37.230$

DIVIDED DIFFERENCES

Definition:

Let the function y = f(x) take the values $f(x_0)$, $f(x_1)$, ……….$f(x_n)$ Corresponding to the values $x_0, x_1 \ldots\ldots x_n$ of the argument x where $x_1-x_0, x_2-x_1, x_3-x_2, \ldots\ldots x_n-x_{n-1}$ need not necessarily be equal.

$$f(x_0,x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$f(x_1,x_2,)=\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$f(x_0,x_1,x_2)=\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$$

$$f(x_1,x_2,x_3)=\frac{f(x_2,x_3)-f(x_1,x_2)}{x_3-x_1}$$ and so on.

1.Find the divided difference table for the following data

x :	**2**	**5**	**10**
y :	**5**	**29**	**109**

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SOLUTION:

Step 1: Given,

x :	**2**	**5**	**10**
y :	**5**	**29**	**109**

Step 2: **To Find the divided difference table for the following data**

Step 3: CALCULATIONS:

The divided difference table

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$
2	5		
		$\frac{29-5}{5-2}=8$	
5	29		$\frac{16-8}{10-2}=1$
		$\frac{109-29}{10-5}=16$	
10	109		

NEWTON'S DIVIDED DIFFERENCE INTERPOLATION

DEFINITION:

The Newton's divided difference interpolation formula is

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + \dots\dots\dots$$
$$+ (x-x_0)(x-x_1)(x-x_2)\dots\dots\dots(x-x_{n-1}) f(x_0,x_1,x_2,\dots\dots\dots x_n)$$

1.Using divided difference formula determine f(3) from the data

x :	**0**	**1**	**2**	**4**	**5**
f(x) :	**1**	**14**	**15**	**5**	**6**

SOLUTION:

STEP 1: Given $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, $x_4 = 5$

$y_0 = 1$, $y_1 = 14$, $y_2 = 15$, $y_3 = 5$, $y_4 = 6$

STEP 2: To determine f(3) from the given data Using divided difference formula

Step 3: REQUIRED FORMULA:

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) +$$
$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0,x_1,x_2,x_3,x_4) \longrightarrow (1)$$

Step 4: CALCULATIONS:

Newton's divided difference table

X	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
0	1				
		$\frac{14-1}{1-0}=13$			
1	14		$\frac{1-13}{2-0}=-6$		
				$\frac{-2+6}{4-0}=1$	
		$\frac{15-14}{2-1}=1$			
2	15		$\frac{-5-1}{4-1}=-2$		$\frac{1-1}{5-0}=0$
		$\frac{5-15}{4-2}=-5$		$\frac{2+2}{5-1}=1$	
4	5		$\frac{1+5}{5-2}=2$		
		$\frac{6-5}{5-4}=1$			
5	6				

Here $x_0=0$, $f(x_0)=1$

$x_1=1$ $f(x_0,x_1)=13$

$x_2=2$, $f(x_0,x_1,x_2)=-6$

$x_3=3$, $f(x_0,x_1,x_2,x_3)=1$

$x_4=4$, $f(x_0,x_1,x_2,x_3,x_4)=0$

Substitute the above values in equation (1)

$$f(x)=0+(x-0)1+(x-0)(x-1)13++(x-0)(x-1)(x-2)(-6)+(x-0)(x-1)(x-2)(x-3)0$$

$$f(x)=-6x^3+21x^2-11x-13 \longrightarrow (2)$$

Substitute x=3 in equation (2)

f (3) = -29

RESULT: $f(x)=-6x^3+21x^2-11x-13$ & f (3) = -29.

2.Using Newton's divided difference formula, find u(3).

given u(1)= -26 , u(2)= 12,u(4)=256,u(6)=844.

Sol:

We form the divided difference table since the intervals are unequal

x	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)
1	-26			
		$\frac{12+26}{2-1} = 38$		
2	12		$\frac{122-38}{4-1} = 28$	
				$\frac{43-28}{6-1} = 3$
		$\frac{256-12}{4-2} = 122$		
4	256		$\frac{294-122}{6-2} = 43$	
		$\frac{844-256}{6-4} = 294$		
6	844			

By Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots\ldots\ldots$$

Here $u(x) = u(x_0) + (x - x_0) u(x_0, x_1) + (x - x_0)(x - x_1) u(x_0, x_1, x_2) + \ldots\ldots\ldots$

$x_0 = 1$, $x_1 = 2$, $x2 = 4$, $x_3 = 6$

$u(x_0) = -26$, $u(x_0,x_1) = 38$, $u(x_0,x_1,x_2) = 28$, $u(x_0,x_1,x_2,x_3) = 3$

$\therefore$ u(x) = -26+(x-1)38+(x-1)(x-2)28+(x-1)(x-2)(x-3)3

u(3) = -26+(2)(38)+(2)(1)(28)+(2)(1)(-1)(3)

= -26 + 76 + 56 – 6 = 100

3. Find f(x) as a polynomial in x for the following data by Newton's divided difference formula.

x	-4	-1	0	2	5
f(x)	1245	33	5	9	1335

Sol:

X	f(x)	Δ f(x)	Δ^2 f(x)	Δ^3 f(x)	Δ^4 f(x)
- 4	1245				
		$\frac{33-1245}{(-1)-(-4)}=-404$			
-1	33		$\frac{-28-(-404)}{0-(-4)}=94$		
		$\frac{5-33}{0-(-1)}=-28$		$\frac{10-94}{2-(-4)}=-14$	
0	5		$\frac{2-(-28)}{2-(-1)}=10$		$\frac{13+14}{5-(-4)}=3$
		$\frac{9-5}{2-0}=2$		$\frac{88-10}{5-(-1)}=13$	
2	9		$\frac{442-2}{5-0}=88$		
		$\frac{1335-9}{5-2}=442$			
5	1335				

By Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) +$$
$$(x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) +$$
$$(x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0,x_1,x_2,x_3,x_4)$$

Here $x_0 = -4$, $x_1 = -1$, $x_2 = 0$, $x_3 = 2$, $x_4 = 5$

$f(x_0) = 1245$, $f(x_0, x_1) = -404$, $f(x_0, x_1, x_2) = 94$, $f(x_0, x_1, x_2, x_3) = -14$

$f(x_0, x_1, x_2, x_3, x_4) = 3$

$f(x) = 1245+(x+4)(-404)+(x+4)(x+1)(94)+$

$(x+4)(x+1)(x)(-14)+(x+4)(x+1)(x)(x-2)(3)$

$= 1245-404x-1616+(94)(x^2+5x+4)-14x(x^2+5x+4)+3x[(x^2+5x+4)(x-2)]$

$= -14x^3+24x^2+10x+5+3x^4+15x^3-24x^2-24x$

$= 3x^4+x^3-14x+5$

NEWTON FORWARD AND BACKWARD DIFFERENCE FORMULA

DEFINITION: (Newton forward difference table)

Let the function y=f(x) take the values $y_0, y_1, ...y_n$ at the points $x_0, x_1, ...x_n$

X	Y	ΔY	Δ^2Y	Δ^3Y
x_0	y_0	$y_1 - y_0 (\Delta y_0)$		
			$\Delta y_1 - \Delta y_0 (\Delta^2 y_0)$	
x_1	y_1	$y_2 - y_1 (\Delta y_1)$		$\Delta^2 y_1 - \Delta^2 y_0 (\Delta^3 y_0)$
			$\Delta y_2 - \Delta y_1 (\Delta^2 y_1)$	
x_2	y_2	$y_3 - y_2 (\Delta y_2)$		$\Delta^2 y_2 - \Delta^2 y_1 (\Delta^3 y_1)$
			$\Delta y_3 - \Delta y_2 (\Delta^2 y_2)$	
x_3	y_3	$y_4 - y_3 (\Delta y_3)$		
…..	…..	……..		

DEFINITION: (Newton forward difference interpolation formula)

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ... + \frac{u(u-1)(u-2)......u-(n-1)}{n!}\Delta^n y_0$$

Where $u = \dfrac{x - x_0}{h}$

DEFINITION: (Newton Backward difference interpolation formula)

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + ... + \frac{v(v+1)(v+2)....v+(n-1)}{n!}\nabla^n y_n .$$

Where $v = \dfrac{x - x_n}{h}$

PROBLEMS:

1. Using Newton's forward interpolation formula, find the polynomial f(x) satisfying the following data. Hence evaluate y at x = 5.

x	4	6	8	10
y	1	3	8	10

SOLUTION:

STEP 1: Given $x_0 = 4$, $x_1 = 6$, $x_2 = 8$, $x_3 = 10$

$y_0 = 1$, $y_1 = 3$, $y_2 = 8$, $y_3 = 10$

STEP 2: To find the polynomial f(x) from the following data Using Newton's forward interpolation

formula. Hence evaluate y at x = 5.

STEP 3: REQUIRED FORMULA:

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \quad \text{where } u = \frac{x - x_0}{h}$$

STEP 4: CALCULATIONS

Difference Table :

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
(x_0) 4	(y_0) 1			
		3-1 = 2(Δy_0)		
(x_1) 6	(y_1) 3		5-2 =3($\Delta^2 y_0$)	
		8-3 =5 (Δy_1)		-3-3 = -6($\Delta^3 y_0$)
(x_2) 8	(y_2) 8		2-5 =-3($\Delta^2 y_1$)	
		10-8 = 2(Δy_2)		
(x_3) 10	(y_3) 10			

There are only 4 data given. Hence the polynomial will be degree 3

Here $x_0 = 0$, h=5- 0 = 5[difference]

$$u = \frac{x-0}{5} = \frac{x}{5}$$

$$y(x) = P_3(x) = 14 + \frac{\left(\frac{x}{5}\right)}{1!}(365) + \frac{\left(\frac{x}{5}\right)\left(\frac{x}{5}-1\right)}{2!}(700) + \frac{\left(\frac{x}{5}\right)\left(\frac{x}{5}-1\right)\left(\frac{x}{5}-2\right)}{3!}(375)$$

$$= 14 + x(73) + x(x-5)(14) + \frac{1}{6}x(x-5)(x-10)(3)$$

$$= 14 + 73x + 14[x^2 - 5x] + \frac{1}{2}x[x^2 - 15x + 50]$$

$$= 14x^2 + 3x + 14 + \frac{1}{2}[x^3 - 15x^2 + 50x]$$

$$= \frac{1}{2}[x^3 + 13x^2 + 56x + 28]$$

$$y(2) = P_3(2) = \frac{1}{2}[2^3 + 13(2)^2 + 56(2) + 28]$$

$$= \frac{1}{2}[200] = 10$$

2. Use Newton's backward interpolation formula to construct an interpolating polynomial of degree 3 for the data:
f(-0.75) = -0.07181250, f(-0.5) = -0.024750, f(-0.25) = 0.33493750,
f(0) = 1.10100. Hence find f($-\frac{1}{3}$).

Sol:

We form the difference table

x	Y	$\triangle y$	$\triangle^2 y$	$\triangle^3 y$
(x_0) -0.75	(y_0) - 0.07181250			
		0.0470625		
(x_1) -0.5	(y_1) - 0.024750		0.312625	
		0.3596875		($\nabla^3 y_3$)0.09375
(x_2) -0.25	(y_2) 0.33493750		($\nabla^2 y_3$)0.400375	
		(∇y_3)0.7660625		
(x_3) 0	(y_3) 1.10100			

Newton's backward difference formula is

$$y(x) = y_3 + \frac{v}{1!}\nabla y_3 + \frac{v(v+1)}{2!}\nabla^2 y_3 + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_3$$

where v = $\frac{x - x_3}{h}$

Here x_3 = 0, h= 0.25, v = $\frac{x}{0.25} = 4x$

Y(x) = 1.10100 + 4x(0.7660625) + $\frac{4x(4x+1)}{2}$(0.406375) +

$\frac{4x(4x+1)(4x+2)}{6}$(0.09375)

= 1.101+3.06425x+ 3.251x^2 +0.81275x+x^3 +0.75x^2 +0.125x

= x^3 +4.001 x^2 +4.002x+1.101

To find f($-\frac{1}{3}$)

y($-\frac{1}{3}$) = $(-\frac{1}{3})^3$ +4.001$(-\frac{1}{3})^2$ +4.002($-\frac{1}{3}$)+1.101

= 0.174518518

NEWTON'S DIFFERENTIATION FORWARD FORMULA

$$y(x) = y_0 + \frac{u}{1}\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\ldots \text{ Where } \quad u = \frac{x - x_0}{h}$$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right)$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left(\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right)$$

PROBLEMS BASED ON NEWTON'S FORWARD DIFFERENCE FORMULA (EQUAL INTERVALS)

1.For the following values of x and y, find the first derivative at $x = 1.05$

$x:$	1	1.05	1.1	1.15	1.2
$y:$	1	1.025	1.049	1.072	1.095

SOLUTION:

STEP 1: Given $x_0 = 1$, $x_1 = 1.05$, $x_2 = 1.1$, $x_3 = 1.15$, $x_4 = 1.2$

$y_0 = 1$, $y_1 = 1.025$, $y_2 = 1.049$, $y_3 = 1.072$, $y_4 = 1.095$

STEP 2: To find Find the first and second derivatives of the function at $x = 0.4$ using Newton's formula.

STEP 3: REQUIRED FORMULA:

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right)$$

STEP 4: CALCULATIONS

Difference Table :

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
(x_0) 1	(y_0) 1				
		(Δy_0)0.25			
(x_1) 1.05	(y_1) 1.025		$(\Delta^2 y_0)$-.001		
				$(\Delta^3 y_0)$0	
(x_2) 1.1	(y_2) 1.049	(Δy_1).024	$(\Delta^2 y_1)$-.001		$(\Delta^4 y_0)$.001
				$(\Delta^3 y_1)$0.001	
(x_3) 1.15	(y_3) 1.072	(Δy_2).023	$(\Delta^2 y_2)$0		
(x_4) 1.2		(Δy_2).023			
	(y_4) 1.095				

There are only5 data given. Hence the polynomial will be degree 4

Here x_0 = 1.05, h=0.05 [difference]

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{.05}\left(0.024 + \frac{1}{2}(.001) + \frac{1}{3}(.001) + \frac{1}{4}(.001)\right) = 0.493$$

=

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{(.05)^2}\left((-.001) - .001 + \frac{11}{12}(.001)\right) = -1.1648$$

RESULT: y'(1.05) = 0.493 & y''(1.05) = −1.1648

2. **Find $\frac{dy}{dx}$ at x = 1 from the following table:**

x	**1**	**2**	**3**	**4**
Y	**1**	**8**	**27**	**64**

Sol: The forward difference table is as follows

x	y	Δ	Δ^2	Δ^3
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		
4	64			

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{\Delta^2}{2} y_0 + \frac{\Delta^3}{3} y_0 + \ldots\ldots\right)$$

$$\left(\frac{dy}{dx}\right)_{x=1} = 3$$

3.Find $y' and y''$ at x=1.25 for the data given:

$x:$	1.00	1.05	1.10	1.15	1.20
$y:$	1.00000	1.02470	1.04881	1.07238	1.09544

SOLUTION:

STEP 1: Given $x_0 = 1$, $x_1 = 1.05$, $x_2 = 1.1$, $x_3 = 1.15$, $x_4 = 1.2$

$y_0 = 1$, $y_1 = 1.025$, $y_2 = 1.049$, $y_3 = 1.072$, $y_4 = 1.095$

STEP 2: To find Find the first and second derivatives of the function at $x = 1.25$ using Newton's

STEP 3: REQUIRED FORMULA:

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left(\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \dots\right)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \dots\right)$$

Difference Table :

X	Y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
(x_0) 1	(y_0) 1				
		(∇y_0)0.25			
(x_1) 1.05	(y_1) 1.025		$(\nabla^2 y_0)$-.001		
				$(\nabla^3 y_0)$0	
(x_2) 1.1	(y_2) 1.049	(∇y_1).024	$(\nabla^2 y_1)$-.001		$(\nabla^4 y_0)$.001
				$(\nabla^3 y_1)$0.001	
(x_3) 1.15	(y_3) 1.072	(∇y_2).023	$(\nabla^2 y_2)$0		
(x_4) 1.2	(y_4) 1.095	(∇y_2).023			

There are only5 data given. Hence the polynomial will be degree 4

Here $x_0 = 1.25$, h=0.05 [difference]

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{.05}\left(0.023 + \frac{1}{2}(0) + \frac{1}{3}(.001) + \frac{1}{4}(.001)\right) = 0.4716$$

=

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{(.05)^2}\left((0) + .001 + \frac{11}{12}(.001)\right) = .764$$

RESULT: y'(1.2) = 0.4716 & y''(1.2) = .764

4. **Find f ′(3) and f′′(3) for one following data:**

X	**3.0**	**3.2**	**3.4**	**3.6**	**3.8**	**4.0**
F(x)	**-14**	**-10.032**	**-5.296**	**-0.256**	**6.672**	**14**

Sol:

Since we require f ′(3) and f′′(3) we use Newton's forward formula.

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3.0	-14					
		3.968				
3.2	-10.032		0.768			
		4.736		-0.464		
3.4	-5.296		0.304		2.048	
		5.04		1.584		-5.12
3.6	-0.256		1.888		-3.072	
		6.928		-1.488		
3.8	6.672		0.4			
		7.328				
4.0	14					

By Newton's forward formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = 9.4665$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = 184.4075$$

5. Find f ′(0) and f′′(4) for one following data:

X	**0**	**1**	**2**	**3**	**4**
F(x)	**1**	**2.718**	**7.381**	**20.086**	**54.598**

Sol:

Since we require f ′(3) and f′′(3) we use Newton's forward formula.

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		1.718			
1	2.718		2.945		
		4.663		5.097	
2	7.381		8.042		8.668
		12.705		13.765	
3	20.086		21.807		
		34.512			
4	54.598				

By Newton's forward formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = -0.2225$$

By Newton's backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = 52.1705$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = 43.5177$$

<u>Formula for Trapezoidal rule</u>

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{2}\left((y_0 + y_n) + 2(y_1 + y_2 + y_3 + \ldots\ldots\ldots\ldots y_{n-1})\right).$$

PROBLEMS BASED ON TRAPEZOIDAL

1.Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with 6 subintervals.

SOLUTION:

STEP1: Given,y(x) = $\int_0^1 \frac{dx}{1+x^2}$ & h=1/6

So,

$x:$	0	1/6	2/6	3/6	4/6	5/6	1
Y	1	36/37	9/10	4/5	9/13	36/61	1/2

STEP2: To Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule

FORMULA FOR SIMPSON'S ONE THIRD RULE

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\left((y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots\ldots\ldots y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots\ldots\ldots y_{n-2})\right).$$

2.Find the value of $\log_e 2^{\frac{1}{3}}$ from $\int_0^1 \frac{x^2dx}{1+x^3}$ using Simpson's $\frac{1}{3}$ rule with h=0.25.

SOLUTION:

STEP1: Given,y(x) = $\int_0^1 \frac{x^2dx}{1+x^3}$ & h=0.25.

So,

$x:$	0	.25	.5	.75	1
Y	0	.06154	.2222	.3956	.5

STEP2: To Evaluate Find the value of $\log_e 2^{\frac{1}{3}}$ from $\int_0^1 \frac{x^2 dx}{1+x^3}$ using Simpson's $\frac{1}{3}$ rule

STEP3: Required formula:

$$\int_{x_0}^{x_0+nh} y(x)dx = \frac{h}{3}\left((y_0 + y_n) + 4(y_1 + y_3 + ...y_{n-1}) + 2(y_2 + y_4 + ...y_{n-2})\right).$$

STEP4: calculations:

$$\int_0^1 y(x)dx = \frac{.25}{3}(0 + .5 + 2(.2222) + 4(.06154 + 0.3956)).$$

=0.231046

$$\int_0^1 \frac{x^2 dx}{1+x^3} = \frac{1}{3}\log(1+x^2)$$

$$\log(1+x^2) = 0.231046$$

RESULT: $\int_0^1 \frac{x^2 dx}{1+x^3} = \frac{1}{3}\log(1+x^2)$: $\log(1+x^2) = 0.231046$

3. **Using Trapezoidal rule , Evaluate** $\int_{-1}^{1} \frac{dx}{1+x^2}$ **taking 8 intervals.**

Sol:

Here y(x) = $\frac{1}{1+x^2}$

Length of the interval = 2

So we divided 8 equal interval with h = 0.25

We form the table

X	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
Y	0.5	0.64	0.8	0.9412	1	0.9412	0.8	0.64	0.5

By Trapezoidal Rule

$$\int_{-1}^{1} \frac{1}{1+x^2} dx = \frac{h}{2}\left[\begin{array}{l} Sum\,of\ the\ first\ and\ last\ ordinate + \\ 2(Sum\,of\ the\, remaining\ ordinates) \end{array}\right]$$

= $\frac{0.25}{2}$ [(0.5+0.5)+2(0.64+0.8+0.9412+0.8+0.64)]

= $\frac{0.25}{2}$ [1+2(5.7624)]

= $\frac{0.25}{2}$ [12.5248]

= 1.5656.

4. Dividing the range into 10 equal parts , find the value of $\int_0^{\frac{\pi}{2}} \sin x \quad dx$

by (i) Trapezoidal rule ii)Simpson's rule.

Sol:

X	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
Y= sinx	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9877	1

i) By Trapezoidal Rule

$$\int_0^{\pi} \sin x\ dx = \frac{h}{2}[(y_0 + y_{11}) + 2(y_1 + y_2 + \ldots\ldots + y_{10})]$$

$$= 0.9980$$

ii) By Simpson's Rule

$$\int_0^{\frac{\pi}{2}} \sin x\ \ dx = \frac{h}{2}\left[(y_0 + y_{11}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + \ldots + y_{10}\right]$$

$$= 1.0000$$

5. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by (i) Trapezoidal rule (ii) Simpson's Rule .Also check up the result by actual integration.

Sol:

Here b – a = 6 – 0 = 6.

Divide into 6 equal parts, h = 1

X	0	1	2	3	4	5	6
F(x)= $\frac{1}{1+x^2}$	1.00	0.500	0.200	0.100	0.058824	0.038462	0.27027

There are 7 ordinate's (n = 6) .We use the formula

i) Trapezoidal Rule:

I = $\int_0^6 \frac{dx}{1+x^2}$ = 1/2 [(1+0.027027)+2(0.5+0.2+0.1+0.058824+0.038462)]

= 1.41079950.

ii) By Simpson's One – Third Rule:

I = 1 / 3[(1+0.027027)+2(0.2+0.058824)+4(0.5+0.1+0.038462)]

= 1/3 (1.027027+0.517648+2.553848)

= 1.36617433.

iii) By Simpson's 3 / 8 Rule:

I = 3 / 8 [(1+0.027027)+3(0.5+0.2+0.058824+0.038462)+2*0.1]

= 1.35708188.

iv) The actual integration

$$I = I = \int_0^6 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^6$$

$$= 1.40564765$$

6. Evaluate $\int_0^5 \frac{dx}{4x+5}$ by simpson's 1/3 rule .Hence find the value of $\log_e 5$ (n = 10)

Sol:

Here $y(x) = \frac{1}{4x+5}$, **h** $=1/2$

X	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
Y(x)	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476	0.0434	0.04

By simpson's 1 / 3 rule :

$$\int_0^5 \frac{dx}{4x+5} = \frac{1}{6}\begin{bmatrix}(0.2+0.04)+(0.1111+0.0769+0.0588+0.0476)+\\ 4(0.1429+0.0909+0.0667+0.0526+0.0434)\end{bmatrix}$$

$= 0.4045.$ …………………(1)

$$\int_0^5 \frac{dx}{4x+5} = \left[\frac{\log(4x+5)}{4}\right]_0^5 = \frac{1}{4}\log 5$$ **……………………(2)**

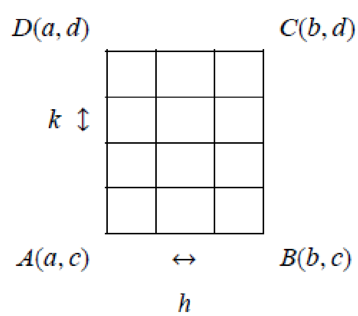
from 1 and 2

$$\frac{1}{4}\log 5 = 0.4025$$

$$\log 5 = 1.61.$$

DOUBLE INTEGRALS BY TRAPEZOIDAL

$$\text{i.e., } I = \frac{hk}{4}\left\{\begin{array}{l}[\text{sum of values of } f \text{ at the four corners}]\\ +2\begin{bmatrix}\text{sum of values of } f \text{ at the nodes}\\ \text{on the boundary except the corners}\end{bmatrix}\\ +4\,[\text{sum of the values at the interior nodes}]\end{array}\right\}$$



where $h = \frac{b-a}{n}$, $k = \frac{d-c}{m}$

where $n =$ is number of equal intervals in (a, b).

where $m =$ is number of equal intervals in (c, d).

PROBLEM 01. **Evaluate the integral $\int_1^{1.4}\int_2^{2.4}\frac{dx\,dy}{xy}$ using Trapezoidal rule. Verify your results by actual integration.**

Solution: $f(x,y)=\frac{1}{xy}$, x varies from $(2, 2.4)$

y varies from $(1, 1.4)$

Divide the range of x and y into 4 equal parts.

$$h=\frac{2.4-2}{4}=0.1,\ k=\frac{1.4-1}{4}=0.1$$

The values of $f(x,y)$ at the nodal points are given in the table :

y \ x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3698	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

By Trapezoidal rule for double integration

$$I=\frac{hk}{4}\left[\begin{array}{l}\text{Sum of values of } f \text{ at the four corners}\\ +2\left(\begin{array}{l}\text{Sum of values of } f \text{ at the nodes}\\ \text{on the boundary except the corners}\end{array}\right)\\ +4\,(\text{Sum of the values at the interior nodes})\end{array}\right]$$

$$=0.0614$$

By actual integration

$$\int_1^{1.4}\int_2^{2.4}\frac{1}{xy}dx\,dy=\int_1^{1.4}\left(\int_2^{2.4}\frac{1}{x}dx\right)\frac{1}{y}dy=\int_1^{1.4}(\log x)_2^{2.4}\,\frac{1}{y}dy$$

$$=(\log 2.4-\log 2)\,(\log y)_1^{1.4}$$

$$=0.0613$$

PROBLEM 01 **Evaluate the integral** $\int_1^{1.4}\int_2^{2.4}\frac{dx\,dy}{xy}$ **using Simpson's rule. Verify your results by actual integration.**

Solution: $f(x,y)=\frac{1}{xy}$, x varies from $(1, 1.4)$

y varies from $(2, 2.4)$

Divide the range of x and y into 4 equal parts.

$$h=\frac{2.4-2}{4}=0.1,\ k=\frac{1.4-1}{4}=0.1$$

The values of $f(x,y)$ at the nodal points are given in the table :

y \ x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3698	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

$$=\frac{(0.1)(0.1)}{9}\left[\begin{array}{l}(0.5+0.4167+0.2976+0.3571)\\+2(0.4545+0.3472+0.3247+0.4167)\\+4\left(\begin{array}{c}0.4762+0.4348+0.3788+0.3205+0.3106\\+0.3401+0.3846+0.4545\end{array}\right)\\+4(0.3788)\\+8(0.3968+0.3623)\\+8(0.3497+0.4132)\\+16(0.3663+0.3344+0.4329+0.3953)\end{array}\right]$$

$$=\frac{0.01}{9}(55.2116)=0.0613$$

UNIT-V
NUMERICAL SOLUTION OF ORDINARY IDFFERENTIAL EQUATIONS

1.State the disadvantage of Taylor series method.

Solution:

In the differential equation $\frac{dy}{dx} = f(x, y)$ the function $f(x, y)$ may have a complicated algebraically structure. Then the evaluation of higher order derivatives may become tedious. This is the demerit of this method.

2.Write down the fourth order Taylor's series Algorithm.

Solution:

$$y_{m+1} = y_m + h\, y_m' + \frac{h^2}{2!} y_m'' + \frac{h^3}{3!} y_m''' + \frac{h^4}{4!} y_m^{iv}$$

Here y_m^{n} denotes the r^{th} derivative of y w.r.to x at the point (x_m, y_m).

3. Using Taylor series method find $y(1.1)$ given that $y' = x + y, y(1) = 0$.

Solution: Given $y' = x + y, y(x = 1) = 0 \Rightarrow x_0 = 1, y_0 = 0, x_1 = 1.1, h = x_1 - x_0 = 1.1 - 1 = 0.1$

$\therefore$ Taylor's series formula is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \cdots \quad (1)$$

$y' = x + y$	$y_0' = x_0 + y_{0(x_0=1, y_0=0)} = 1 + 0 = 1$ (2)
$y'' = 1 + y'$	$y_0'' = 1 + y_0' = 1 + 1 = 2$ [by (2)] (3)
$y''' = 0 + y'' = y''$	$y_0''' = y_0'' = 2$ [by (3)] (4)
$y^{iv} = y'''$	$y_0^{iv} = y_0''' = 2$ [by (4)]

$$\therefore (1) \Rightarrow y_1 = 0 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2)$$

$$\text{i.e., } y(1.1) = (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} + \cdots$$

$$= 0.1103083$$

$$\equiv 0.11031$$

4. Find $y(0.1)$ if $\frac{dy}{dx} = 1 + y, y(0) = 1$ using Taylor series method.

Solution: Given $y' = 1 + y, y(x = 0) = 1 \Rightarrow x_0 = 0, y_0 = 1, x_1 = 0.1, h = x_1 - x_0 = 0.1 - 0 = 0.1$

$\therefore$ Taylor's series formula is

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{iv}_0 + \cdots \quad (1)$$

$y' = 1 + y$	$y'_0 = 1 + y_{0(x_0=0, y_0=1)} = 1 + 1 = 2$		(2)
$y'' = 0 + y' = y'$	$y''_0 = y'_0 = 2$	[by (2)]	(3)
$y''' = y''$	$y'''_0 = y''_0 = 2$	[by (3)]	(4)
$y^{iv} = y'''$	$y^{iv}_0 = y'''_0 = 2$	[by (4)]	

$$\therefore (1) \Rightarrow y_1 = 1 + \frac{0.1}{1!}(2) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2)$$

i.e., $y(1.1) = 1.210341667$

$\equiv 1.21034$

5. Write down the Euler algorithm to the differential equation $\frac{dy}{dx} = f(x, y)$.

Solution:

$y_{n+1} = y_n + h\, f(x_n, y_n)$ When n = 0, 1, 2...

This is Euler algorithm. It can also be written as $y(x+h) = y(x) + h\, f(x, y)$

1. Find $y(0.1)$ by using Euler's method given that $\frac{dy}{dx} = x + y, y(0) = 1$.

Solution : Given, $\frac{dy}{dx} = x + y = f(x, y), x_0 = 0, y_0 = 1, x_1 = 0.1, h = 0.1$

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + (0.1)[x_0 + y_0]$$
$$= 1 + (0.1)[0 + 1]$$
$$= 1 + 0.1$$
$$= 1.01$$

i.e., $y(0.1) = 1.01$

2.The equation Solve $y' = 1 - y, y(0) = 0$ by modified Euler method.

Solution: Given $y' = f(x, y) = 1 - y, x_0 = 0, y_0 = 0.$

Let $h = 0.1, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$

We have to find y_1, y_2, y_3.

By Modified Euler's method:

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{1}{2}hf(x_n, y_n)\right]$$

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right]$$ [Here $f(x_0, y_0) = f(x = x_0, y = y_0) = 1 - y_0 = 1 - 0 = 1$]

$$\therefore y_1 = 0 + (0.1)f\left[0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}(1)\right]$$

$$= 0.1 f[0.05, 0.05]$$

$$= \{(0.1)[1 - y]\}_{\binom{x=0.05}{y=0.05}} = 1 - 0.05$$

$$y_1 = 0.095$$

$$\text{Now } y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2}f(x_1, y_1)\right]$$ [Here $f(x_1, y_1) = 1 - y_1 = 0.905$]

$$\text{i.e., } y_2 = 0.905 + (0.1)f\left[0.1 + \frac{0.1}{2}, 0.095 + \frac{0.1}{2}f(0.1, 0.095)\right]$$

$$y_2 = 0.18098$$

$$\text{Now } y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2}f(x_2, y_2)\right]$$ [Here $f(x_2, y_2) = 1 - y_2 = 1 - 0.18098 = 0.81902$]

$$\text{i.e., } y_3 = 0.18098 + (0.1)f\left[0.2 + \frac{0.1}{2}, 0.18098 + \frac{0.1}{2}f(0.2, 0.18098)\right]$$

$$y_3 = 0.258787$$

$$\boxed{\therefore y_0 = 0, y_1 = 0.095, y_2 = 0.18098, y_3 = 0.258787}$$

The Runge-Kutta formula of fourth order to solve $\frac{dy}{dx} = f(x, y)$ **with** $y(x_o) = y_o$

Solution:

Let h denotes the interval between equidistant values of x . If the initial values are (x_o, y_o) the first increment in y is computed from the formulae's.

$$K_1 = h\,f(x_o, y_o)$$

$$K_2 = h\,f\left(x_o + \frac{h}{2}, y_o + \frac{K_1}{2}\right)$$

$$K_3 = h\,f\left(x_o + \frac{h}{2}, y_o + \frac{K_2}{2}\right)$$

$$K_4 = h\,f(x_o + h, y_o + K_3)$$

$$and \quad \Delta y = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$Then\ y_1 = y_o + \Delta y$$

The increment in y in the second interval is computed in a similar manner using the same four formulas, using the values x_1, y_1 in the place of x_0, y_0 respectively.

Special advantages of Runge-Kutta method over Taylor series method .

Solution:

Runge-Kutta methods do not require prior calculation of higher derivatives of y(x), as The Taylor's method does. Since the differential equations using in application are often complicated, the calculation of derivatives may be difficult.

Also, the Runge –Kutta formulas involve the computations of f(x,y) at various positions , instead of derivatives and this function occurs in the given equations .

The system is Given $\frac{dy}{dx} = x + y^2, y(0) = 1$**, find** $y(0.1)$ **&** $y(0.2)$ **by Runge-Kutta method for IV**

Solution: Given $y' = f(x, y) = x + y^2, h = 0.1$

$$\& x_0 = 0, y(x_0 = 0) = 1$$

To find $y(0.1) = y_1$

$$k_1 = hf[x_0, y_0] = (0.1) f(0.1) = (0.1)\left[0 + 1^2\right] = 0.1$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = (0.1) f[0 + 0.05, (1 + 0.05)] = (0.1)\left[0.05 + (1.05)^2\right] = 0.11525$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = (0.1) f\left[0.05, 1 + \frac{0.11525}{2}\right] = (0.1)\left[0.05 + (1.057625)^2\right] = 0.116857$$

$$k_4 = hf[x_0 + h, y_0 + k_3] = (0.1) f[0.05, 1 + 0.116857] = (0.1)\left[0.05 + (1.116857)^2\right] = 0.134737$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = \frac{1}{6}[(0.1) + 2(0.1) + 2(0.11525) + (0.116857)]$$

$$= 0.11649$$

$$\therefore y(0.1) = y_1 = y_0 + \Delta y = 1 + 0.11649 = 1.11649 \Rightarrow y_1 = 1.11649$$

To find $y(0.2) = y_2$

$$k_1 = hf[x_1, y_1] = (0.1) f[0.1, 1.11649] = (0.1)[0.1 + (1.116492)] = 0.1347$$

$$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right] = (0.1) f\left[0.1 + 0.05, 1.11649 + \frac{0.1347}{2}\right] = (0.1) f[0.15, 1.18385] = 0.1552$$

$$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right] = (0.1) f\left[0.1 + 0.05, 1.11649 + \frac{0.1552}{2}\right] = (0.1) f[0.15, 1.1941] = 0.1576$$

$$k_4 = hf[x_1 + h, y_1 + k_3] = (0.1) f[0.1 + 0.1, 1.11649 + 0.1576] = 0.18233$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.1571$$

$$y_2 = y_1 + \Delta y = 1.11649 + 0.1571 = 1.27359$$

i.e.,

x	0	0.1	0.2
y	1	1.11649	1.27359

MILNE'S PREDICTOR AND CORRECTOR FORMULA.

Milne's predictor formula is,

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

Milne's corrector formula is,

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

The error term in Milne's corrector formula

The error term is $-\frac{h}{90}\Delta^4 y'_o$.

The error term in Milne's predictor formula ?

The error term is $\frac{14h}{45}\Delta^4\ y'_o$

The equation Given $\frac{dy}{dx} = \frac{1}{2}[x+y], y(0) = 2, y(0.5) = 2.636, y(1) = 3.595, y(1.5) = 4.968$ by Milne's method to find $y(2)$.

Solution: Given

$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$	$x_4 = 2$
$y_0 = 2$	$y_1 = 2.636$	$y_2 = 3.595$	$y_3 = 4.968$	$y_4 = ?$

Here $h = 0.5$ Given $y' = \frac{1}{2}[x+y] = f(x,y)$ By Milne's predictor formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$\text{when } n = 3, y_{3+1,p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \qquad (1)$$

$$\text{Now, } y'_1 = \frac{1}{2}[x_1 + y_1] = \frac{1}{2}[0.5 + 2.636] = 1.568$$

$$y'_2 = \frac{1}{2}[x_2 + y_2] = \frac{1}{2}[1 + 3.595] = 2.2975$$

$$y'_3 = \frac{1}{2}[x_3 + y_3] = \frac{1}{2}[1.5 + 4.968] = 3.234$$

$$(1) \Rightarrow y_{4,p} = 2 + \frac{4(0.5)}{3}[2(1.568) - 2.2975 + 2(3.234)]$$

$$= 6.871\ (= y_4, \text{say})$$

By Milne's corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

$$\text{when } n = 3, y_{4,c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

$$\text{Now, } y'_4 = \frac{1}{2}[x_4 + y_4] = \frac{1}{2}[2 + 6.871] = 4.4355$$

$$(2) \Rightarrow y_{4,c} = 3.595 + \frac{0.5}{3}[2.2975 + 4(3.234) + 4.4355]$$

$$= 3.595 + \frac{0.5}{3}(19.669) = 6.8732$$

$\therefore$ Corrected value of y at 2 is $y(2) = 6.8732$.

THE ADAMS-BASHFORTH PREDICTOR & CORRECTOR FORMULA.

Adam's predictor& corrector formula are

$$y_{n+1,\ p} = y_n + \frac{h}{24}\left(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right)$$

$$y_{n+1,\ c} = y_n + \frac{h}{24}\left(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right)$$

Compare Runge-Kutta methods and Predictor – Corrector methods for solutionof initial value problem.

Runge-Kutta methods:

1. Runge-methods are self-starting method , since they do not use information From previsiously calculated points.

2. As means are self starting an easy change in the step size can be made at any Stage.

3. Since the methods require several evaluations of the function f (x, y), they are time Consuming.

4. In these methods, it is not possible to get any information about truncation error.

Predictor – Corrector methods

1. These methods require information about prior points and so they are not self starting.

2. In these methods, it is possible to get easily a good estimate of truncation error.

Predictor – Corrector method of solving a differential equation

Predictor – Corrector methods are method whish require the value of y at $x_n, x_{n-1}, x_{n-2}, \ldots$ for computing the value of y at x_{n+1}. We first use a formula to find the values of y at x_{n+1} and this is known as a predictor formula . The vakue of y so got is improved or corrected by another formula known as corrector formula.

Adam's method	Error
Predictor	$\frac{251}{720}h^5 f^{(iv)}(\xi)$
Corrector	$\frac{-19}{720}h^5 f^{(iv)}(\xi)$

The method is self – starting and Not self – starting.

	Self-starting method	Not Self-starting method
1.	To find (x_{n+1}, y_{n+1}) we use only the information at (x_n, y_n)	We need past values , based on that only we get the next value.
2.	Example Taylor's series , Euler	Milne's , Adam's method

The equation is Given $\frac{dy}{dx} = x^2(1+y)$, $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$, evaluate $y(1.4)$ by Adam's-Basforth method.

Solution: Given

$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$	$x_4 = 1.4$
$y_0 = 1$	$y_1 = 1.233$	$y_2 = 1.548$	$y_3 = 1.979$	$y_4 = ?$

Here $h = 0.1$

By Adam's predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}\right]$$

$$\text{when } n = 3, y_{4,p} = y_3 + \frac{h}{24}\left[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0\right] \quad (1)$$

Here $y'_0 = x_0^2(1+y_0) = 1^2[1+1] = 2$

$y'_1 = x_1^2(1+y_1) = (1.1)^2[1+1.233] = 2.70193$

$y'_2 = x_2^2(1+y_2) = (1.2)^2[1+1.548] = 3.66912$

$y'_3 = x_3^2(1+y_3) = (1.3)^2[1+1.979] = 5.0345$

$$(1) \Rightarrow y_{4,p} = 1.979 + \frac{0.1}{24}[55(5.0345) - 59(3.66912) + 37(2.70193) - 9(2)]$$

$$= 2.5723$$

By Adam's corrector method,

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}\right]$$

$$(\text{when } n = 3), y_{4,c} = y_3 + \frac{h}{24}\left[9y'_4 + 19y'_3 - 5y'_2 + y'_1\right] \quad (2)$$

$$y'_4 = (x_4)^2(1+y_4) = (1.4)^2[1+2.5871] = 7.0017$$

$$\therefore (2) \Rightarrow y_{4,c} = 1.979 + \frac{0.1}{24}[9(7.030716) + 19(5.0345) - 5(3.60912) + (2.70193)]$$

$$= 2.5749$$