STATISTICS AND NUMERICAL METHODS

MA3251 - SNM - DOWNLOADED FROM STUCOR APP

REGULATION 2021

As per Anna University Syllabus

(MECHANICAL BRANCH)

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Pass % = 100 %

Continuous
Updated
Practice
(CUP)

ALL THE BEST

MA8452   **STATISTICS AND NUMERICAL METHODS**   **L T P C**
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OBJECTIVES:

- This course aims at providing the necessary basic concepts of a few statistical and numerical methods and give procedures for solving numerically different kinds of problems occurring in engineering and technology.

- To acquaint the knowledge of testing of hypothesis for small and large samples which plays an important role in real life problems.

- To introduce the basic concepts of solving algebraic and transcendental equations.

- To introduce the numerical techniques of interpolation in various intervals and numerical techniques of differentiation and integration which plays an important role in engineering and technology disciplines.

- To acquaint the knowledge of various techniques and methods of solving ordinary differential equations.

UNIT I   TESTING OF HYPOTHESIS   **12**

Sampling distributions - Estimation of parameters - Statistical hypothesis - Large sample tests based on Normal distribution for single mean and difference of means -Tests based on t, Chi-square and F distributions for mean, variance and proportion - Contingency table (test for independent) - Goodness of fit.

UNIT II   DESIGN OF EXPERIMENTS   **12**

One way and two way classifications - Completely randomized design - Randomized block design - Latin square design - 2^2 factorial design.

UNIT III   SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS   **12**

Solution of algebraic and transcendental equations - Fixed point iteration method - Newton Raphson method - Solution of linear system of equations - Gauss elimination method - Pivoting - Gauss Jordan method - Iterative methods of Gauss Jacobi and Gauss Seidel - Eigenvalues of a matrix by Power method and Jacobi's method for symmetric matrices.

UNIT IV   INTERPOLATION, NUMERICAL DIFFERENTIATION AND NUMERICAL INTEGRATION   **12**

Lagrange's and Newton's divided difference interpolations - Newton's forward and backward difference interpolation - Approximation of derivates using interpolation polynomials - Numerical single and double integrations using Trapezoidal and Simpson's 1/3 rules.

UNIT V   NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS   12

Single step methods : Taylor's series method - Euler's method - Modified Euler's method - Fourth order Runge-Kutta method for solving first order equations - Multi step methods : Milne's and Adams - Bash forth predictor corrector methods for solving first order equations.

TOTAL : 60 PERIODS

OUTCOMES:

Upon successful completion of the course, students will be able to:

- Apply the concept of testing of hypothesis for small and large samples in real life problems.

- Apply the basic concepts of classifications of design of experiments in the field of agriculture.

- Appreciate the numerical techniques of interpolation in various intervals and apply the numerical techniques of differentiation and integration for engineering problems.

- Understand the knowledge of various techniques and methods for solving first and second order ordinary differential equations.

- Solve the partial and ordinary differential equations with initial and boundary conditions by using certain techniques with engineering applications

TEXT BOOKS:

1. Grewal. B.S. and Grewal. J.S., "Numerical Methods in Engineering and Science", 10 th Edition, Khanna Publishers, New Delhi, 2015.

2. Johnson, R.A., Miller, I and Freund J., "Miller and Freund's Probability and Statistics for Engineers", Pearson Education, Asia, 8 th Edition, 2015.

REFERENCES:

1. Burden, R.L and Faires, J.D, "Numerical Analysis", 9 th Edition, Cengage Learning, 2016.

2. Devore. J.L., "Probability and Statistics for Engineering and the Sciences", Cengage Learning, New Delhi, 8 th Edition, 2014.

3. Gerald. C.F. and Wheatley. P.O. "Applied Numerical Analysis" Pearson Education, Asia, New Delhi, 2006.

4. Spiegel. M.R., Schiller. J. and Srinivasan. R.A., "Schaum's Outlines on Probability and Statistics ", Tata McGraw Hill Edition, 2004.

5. Walpole. R.E., Myers. R.H., Myers. S.L. and Ye. K., "Probability and Statistics for Engineers and Scientists", 8 th Edition, Pearson Education, Asia, 2007.

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Contents

STATISTICS

UNIT- I   TESTING OF HYPOTHESIS

Population:

A population consists of collection of individual units, which may be person's or experimental outcomes, whose characteristics are to be studied.

Sample:

A sample is proportion of the population that is studied to learn about the characteristics of the population.

Random sample:

A random sample is one in which each item of a population has an equal chance of being selected.

Sampling:

The process of drawing a sample from a population is called sampling.

Sample size:

The number of items selected in a sample is called the sample size and it is denoted by 'n'. If $n \geq 30,$ the sample is called large sample and if $n \leq 30,$ it is called small sample

Sampling distribution:

Consider all possible samples of size' n' drawn from a given population at random. We calculate mean values of these samples.

If we group these different means according to their frequencies, the frequency distribution so formed is called sampling distribution.

The statistic is itself a random variate. Its probability distribution is often called sampling distribution.

All possible samples of given size are taken from the population and for each sample, the statistic is calculated. The values of the statistic form its sampling distribution.

Standard error:

The standard deviation of the sampling distribution is called the standard error.

 Notation:

Pop. mean $= \mu;$ Pop. S.D $= \sigma;$ P - Pop. proportion

sample mean = $\bar{x}$; sample S.D = s; P' = sample Proportion

Note

Statistic	S.E (Standard Error)
$\bar{x}$	$\dfrac{\sigma}{\sqrt{n}}$
$p_1' - p_2'$ $\left(\begin{array}{l}\text{Difference of sample}\\ \text{proportions}\end{array}\right)$	$\sqrt{pq\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}$
$\bar{x_1} - \bar{x_2}$ $\left(\begin{array}{l}\text{Difference of sample}\\ \text{means}\end{array}\right)$	$\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}$
p' (Sample proportion)	$\sqrt{\dfrac{pq}{n}}$

Null Hypothesis (H_0)

The hypothesis tested for possible rejection under the assumption that it is true is usually called null hypothesis. The null hypothesis is a hypothesis which reflects no change or no difference. It is usually denoted by H_0

Alternative Hypothesis (H_1)

The Alternative hypothesis is the statement which reflects the situation anticipated to be correct if the null hypothesis is wrong. It is usually denoted by H_1.

For example:

If $H_0 : \mu_1 = \mu_2$ (There is no diff' bet' the means) then the formulated alternative hypothesis is

$H_1 : \mu_1 \neq \mu_2$

ie., either $H_1 : \mu_1 < \mu_2$ (or) $\mu_1 > \mu_2$

Level of significance

It is the probability level below which the null hypothesis is rejected. Generally, 5% and 1% level of significance are used.

Critical Region (or) Region of Rejection

The critical region of a test of statistical hypothesis is that region of the normal curve which corresponds to the rejection of null hypothesis.

The shaded portion in the following figure is the critical region which corresponds to 5% LOS
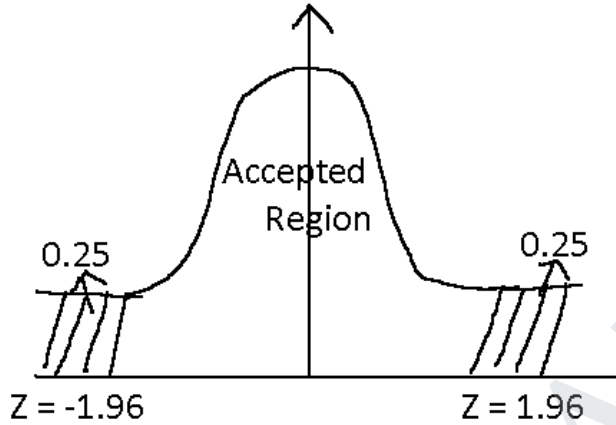
Critical values (or) significant values

The sample values of the statistic beyond which the null hypothesis will be rejected are called critical values or significant values

	Level of significance		
Types of test	1%	5%	10%
Two tailed test	2.58	1.96	1.645
One tailed test	2.33	1.645	1.28

Two tailed test and one-tailed tests:

When two tails of the sampling distribution of the normal curve are used, the relevant test is called two tailed test.

The alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ is taken in two tailed test for $H_0 : \mu_1 = \mu_2$

When only one tail of the sampling distribution of the normal curve is used, the test is described as one tail test $H_1 : \mu_1 < \mu_2$ (or) $\mu_1 > \mu_2$

$$\left. \begin{array}{l} H_0 = \mu_1 = \mu_2 \\ H_1 = \mu_1 \neq \mu_2 \end{array} \right\} \text{ two tailed test}$$

Type I and type II Error

Type I Error : Rejection of null hypothesis when it is correct

Type II Error : Acceptance of null hypothesis when it is wrong

Procedure for testing Hypothesis:

1. Formulate H_0 and H_1
2. Choose the level of significance α
3. Compute the test statistic Z, using the data available in the problem
4. Pick out the critical value at α % level say Z_α
5. Draw conclusion: If $|Z| < Z_\alpha$, accept H_0 at α% level. Otherwise reject H_0 at α% level

Test of Hypothesis (Large Sample Tests)

Large sample tests (Test based in Normal distribution.)

Type - I: (Test of significance of single mean)

Let $\{x_1, x_2, \ldots x_n\}$ be a sample of size $(n \geq 30)$ taken from a population with mean μ and S.D σ. Let $\bar{x}$ be the sample mean. Assume that the population is Normal.

To test whether the difference between Population mean μ and sample mean $\bar{x}$ is significant or not and this sample comes from the normal population whose mean is μ or not.

$H_0 : \mu = $ a specified value

$H_1 : \mu \neq $ a specified value

we choose $\alpha = 0.05(5\%)$ (or) $0.01(1\%)$ as the Level of significance

the test statistic is

$$Z = \frac{\bar{x} - \mu}{S.E(\bar{x})} = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0.1) \text{ for large n.}$$

Note:

1. If σ is not known, for large n, S.E $\bar{x} = \frac{s}{\sqrt{n}}$ where 's' is the sample S.D

Problems:

1. A sample of 900 members is found to have a mean 3.5cm. Can it reasonably regarded as a simple sample from a large population whose mean is 3.38 and a standard deviation 2.4cm?

Solution:

We formulate the null hypothesis that the sample is drawn from population whose mean is 3.38cm.

$$i.e., H_0 : \mu = 3.38$$

$$H_1 : \mu \neq 3.38$$

Hence it is a two-tailed test

Level of significance $\alpha = 0.05$

Test statistic $Z = \dfrac{\overline{x} - \mu}{\dfrac{\sigma}{\sqrt{n}}}$

Given $\overline{x} = 3.5$, $\mu = 3.38$, n = 900, $\sigma = 2.4$

$$\therefore \; Z = \dfrac{3.5 - 3.32}{\dfrac{2.4}{\sqrt{900}}} = 1.5$$

Critical value:

At 5% level, the tabulated value of Z is 1.96

Conclusion:

Since |Z| = 1.5 < 1.96, H_0 is accepted at 5% level of significance

i.e., the sample comes from a population with mean 3.38cm

2. A manufacturer claims that his synthetic fishing line has a mean breaking strength of 8kg and S.D 0.5kg. Can we accept his claim if a random sample of 50 lines yield a mean breaking of 7.8kg. Use 1% level of significance.

Solution:

We formulate $H_0 : \mu = 8$

$$H_1 : \mu \neq 8$$

L.O.S $\alpha = 0.01$

Test statistic $Z = \dfrac{\overline{x} - \mu}{\dfrac{\sigma}{\sqrt{n}}}$

Given $\overline{x} = 7.8$, $\mu = 8$, n = 50, $\sigma = 0.5$

$\therefore\ Z = \dfrac{7.8 - 8}{\dfrac{0.5}{\sqrt{50}}} = -2.828$

$\therefore\ |Z| = 2.828$

Critical value:

At 1% level of significance the table of Z = 2.58

Conclusion:

Since $|Z| > 2.58$, H_0 is rejected at 1% level

i.e., the manufacturer's claim is not accepted

3. A random sample of 200 Employee's at a large corporation showed their average age to be 42.8 years, with a S.D of 6.8 years. Test the hypothesis $H_0 : \mu = 40$ versus $H_1 : \mu > 40$ at $\alpha = 0.01$ level of significance.

Solution:

We set up $\quad H_0 : \mu = 40$

$\quad\quad H_1 : \mu \neq 40$

It is one tailed test.

$\quad\quad$ L.O.S $\alpha = 0.01$

Test statistic $Z = \dfrac{\overline{x} - \mu}{\dfrac{\sigma}{\sqrt{n}}}$

Given $\overline{x} = 42.8$, $\mu = 40$, n = 200, $\sigma = 6.89$

$$\therefore \ Z = \frac{42.8 - 40}{\dfrac{6.89}{\sqrt{200}}} = 5.747$$

Critical value:

For one-tail test, the table value of Z at 1% level = 2.33

Conclusion:

Since |Z| =5.747> 2.33, H_0 is rejected at 1% level.

i.e., The hypothesis $\mu = 40$ is accepted at this level.

Type - II:

Test of significance of difference of two means

Consider two samples of sizes n_1 and n_2 taken from two different populations with population means μ_1 and μ_1 and S.D's σ_1 and σ_2

Let $\overline{x_1}$ and $\overline{x_2}$ be the sample means and S_1 and S_2 be the S.D's of the samples

The formulated null and alternative hypothesis is,

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

The test statistic 'Z' is defined by

$$Z = \frac{\overline{x_1} - \overline{x_2}}{S.E(\overline{x_1} - \overline{x_2})}$$

$$ie., \ Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \sim \ N(0,1)$$

We use the los is $\alpha = 0.05 \ (or) \ 0.01$

If $|Z| < Z_\alpha$, H_0 is accepted at $\alpha \ \%Los$

otherwise, H_0 is rejected at $\alpha \ \%Los$

Note:

In many situations, we do not know the S.D's of the populations (or) population from which the samples are drawn.

In such cases, we can subs the S.D's are of samples S_1 and S_2 in place of σ_1 and σ_2

$\therefore$ The test statistic Z $= \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$

Problems

The mean of two sample large samples of 1000 and 200 members are 67.5 inches and 68 inches respectively. Can the samples be regard as drawn from the population of standard deviation of 2.5 inches? Test at 5% Los

Solution

we set up $H_0 : \mu_1 = \mu_2$

ie., the samples are drawn from the sample population

$H_1: \mu_1 \neq \mu_2$

The test statistic Z $= \dfrac{\overline{x_1} - \overline{x_2}}{\sigma \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

Given $\overline{x_1} = 67.5; \quad n_1 = 1000$

$\overline{x_2} = 68; \quad n_2 = 2000; \quad \sigma = 2.5$

$\therefore \quad Z = \dfrac{67.5 - 68}{2.5\sqrt{\dfrac{1}{1000} + \dfrac{1}{2000}}} = -5.164$

$\therefore \ |Z| = 5.164$

We choose the Los $\alpha = 0.05$

Critical value:

The table values of Z at 5% Los is Z = 1.96

Conclusion:

Since |Z| > 1.96, H_0 is rejected at 5% Los.

∴ The sample cannot be regards as drawn from the same population.

2. Samples of students were drawn from two universities and from the weights is kilogram. The means and S.D's are calculated. Test the significance of the difference between the means of two samples

	Mean	S.D	Sample Size
University A	55	10	400
University B	57	15	100

Solution:

we set up $H_0 : \mu_1 = \mu_2$

ie., there is no significant difference between the sample means

$H_1: \mu_1 \neq \mu_2;$ $\alpha = 0.05$

The test statistic $Z = \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$

Given $\overline{x_1} = 55;$ $s_1 = 10;$ $n_1 = 400$

$\overline{x_2} = 57;$ $s_2 = 15;$ $n_2 = 100$

∴ $Z = \dfrac{55 - 57}{\sqrt{\dfrac{10^2}{400} + \dfrac{15^2}{100}}} = -1.265$

∴ |Z| = 1.265

Critical value:

The table values of Z at 5% Los is Z = 1.96

Conclusion:

Since |Z| < 1.96, H_0 is accepted at 5% Los. We conclude that the difference between the means is not significant.

3. The average hourly wage of a sample of 150 workers is plant A was Rs. 2.56 with a S.D of Rs.1.08. The average wage of a sample of 200 workers in plant B was Rs. 2.87 with a S.D of Rs. 1.28. Can an applicant safety assume that the hourly wages paid by plant B are greater than those paid by plant A?

Solution:

Let x_1 and x_2 denote the hourly wages paid to workers in plant A and plant B respectively.

We set up H_0 : $\mu_1 < \mu_2$ (Plant Bnot greater than Plant A)

H_1 : $\mu_1 < \mu_2$ (one tailed test)

$\alpha = 0.05$

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$$

Given $\overline{x_1} = 2.56$; $s_1 = 1.08$; $n_1 = 150$

$\overline{x_2} = 2.87$; $s_2 = 1.28$; $n_2 = 200$

$$\therefore \quad Z = \frac{2.56 - 2.87}{\sqrt{\dfrac{(1.08)^2}{150} + \dfrac{(1.28)^2}{200}}} = -2.453$$

$\therefore$ |Z| = 2.453

Critical value:

The table values of Z at 5% in case of one-tailed test is Z = 1.645

Conclusion:

Since $|Z| > 1.643$, H_0 is rejected at 5% Los.

$\therefore$ The hourly wage paid by Plant B are greater than those paid by Plant A

4. A sample of size 30 from a normal population yielded 80 and variance 150. A sample of size 40 from a second normal population yielded the sample mean 71 and variance 200. Test $H_0 : \mu_1 - \mu_2 = 2$. Versus $H_1 : \mu_1 > \mu_2 = 2$

Solution:

$H_0 : \mu_1 - \mu_2 = 2$.

ie., the diff 'bet the means of two population is 2

Versus $H_1 : \mu_1 > \mu_2 = 2$ (one tailed)

Test Statistic $Z = \dfrac{\overline{x_1} - \overline{x_2} \quad \mu_1 - \mu_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$

$\therefore Z = \dfrac{(80 - 71) - 2}{\sqrt{\dfrac{150}{30} + \dfrac{200}{40}}} = 2.215$

Critical value:

For one tail test, at 5% Los the table value of $z = 1.645$

Conclusion:

Since $|Z| > 1.645$, H_0 is rejected.

$\therefore$ The formulated null hypothesis $H_0 : \mu_1 - \mu_2 = 2$ is wrong

5. A buyer of electric bulbs purchases 400 bulbs; 200 bulbs of each brand. Upon testing these bulbs be found that brand A has an average of 1225 hrs with a S.D of 42 hrs. where as brand B had a mean life of 1265 hrs with a S.D of 60 hrs. Can the buyer be certain that brand B is Superior than brand A in quality?

Solution:

$$H_0 : \mu_1 = \mu_2;$$

ie., the two brands of bulbs do not differ in quality

ie., they have the same mean life

$H_1: \mu_1 < \mu_2$ (one tailed)

L.o.s : $\alpha = 0.05$

Test Statistic $Z = \dfrac{\overline{x_1} - \overline{x_2}}{S.E \; \overline{x_1} - \overline{x_2}} = \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$

Here, $\overline{x_1} = 1225;$ $s_1 = 42;$ $n_1 = 200$

$\overline{x_2} = 1265;$ $s_2 = 60;$ $n_2 = 200$

$\therefore Z = \dfrac{1225 - 1265}{\sqrt{\dfrac{(42)^2}{200} + \dfrac{(60)^2}{200}}} = \dfrac{-40}{5.18} = -7.72$

$\Rightarrow |Z| = 7.72$

Critical value:

The critical value of Z at 5% Los Z = 1.645.

Conclusion:

Since |z| < 1.645 H_0 is rejected.

$\therefore$ The brand B is superior to brand A in equality.

Type - III:

Test of significance of single proportion:

If 'x' is the number of times possessing a certain attribute in a sample of n items,

The sample proportion $p' = \dfrac{x}{n}$

p' : sample porportion;

p: population proportion.

The hypothesis $H_0 : p = p'$

ie., p has a specified value

Alternative hyp: $H_1 : p \neq p'$

Test statistic $Z = \dfrac{p' - p}{\sqrt{\dfrac{pq}{n}}}$

Since the sample is large $Z \sim N(0,1)$

Problems

1. A coin is tossed 400 times and it turns up head 216 times. Discuss whether the coin may be regarded as unbiased one.

Solution

we set up H_0 : coin is unbiased

ie., $p = \dfrac{1}{2} \Rightarrow q = 1 - p = \dfrac{1}{2}$

H_1 : coin is biased

$\alpha = 0.05$

Test statistic $Z = \dfrac{p' - p}{\sqrt{\dfrac{pq}{n}}}$

Here $p' = \dfrac{216}{400} ; n = 400$

$$\therefore \quad Z = \frac{0.54 - 0.5}{\sqrt{\dfrac{1}{600}}} = 1.6$$

Table value of $Z = 1.96$

Conclusion:

Since $|z| < 1.96$, H_0 is accepted at 5% Los

Hence the coin may be regarded as unbiased

2. In a city of sample of 500 people, 280 are tea drinkers and the rest are coffee drinkers. Can we assume that both coffee and tea are equally popular in this city at 5% Los.

Solution:

we set up $H_0 : p = \dfrac{1}{2}$

ie., the coffee and tea are equally popular

$H_1 : p \neq \dfrac{1}{2}$

Test statistic $Z = \dfrac{p' - p}{\sqrt{\dfrac{pq}{n}}}$

Here $p' = \dfrac{280}{500} = 0.56; n = 500; \ p = 0.5$

$\Rightarrow q = 1 - p = 0.5$

$$\therefore \quad Z = \frac{0.56 - 0.5}{\sqrt{\dfrac{0.5 \times 0.5}{500}}} = 2.68$$

Conclusion:

Since $|z| > 1.96$, H_0 is rejected at 5% level

Both type of drinkers are not popular at 5% Los.

3. A manfacturing company claims that atleast 95% of its products supplied confirm to the specifications out of a sample of 200 prodcuts, 18 are defective. Test the claim at 5% Los.

Solution

we set up H_0 : The proportion of the products confirming to specification is 95%

ie., p = 0.95

H_1 : p < 0.95 (one tailed test)

$$Z = \frac{p' - p}{\sqrt{\dfrac{pq}{n}}}$$

Here $p' = \dfrac{200 - 18}{200} = 0.91; n = 200$

$p = 0.95 \Rightarrow q = 1 - p = 0.05$

$\therefore Z = \dfrac{0.91 - 0.95}{\sqrt{\dfrac{0.95 \times 0.05}{200}}} = -2.595 \Rightarrow |Z| = 2.595$

Critical value : at 5%Los $Z_\alpha = 1.645$

Conclusion:

$|z| = 2.595 > 1.645$, H_0 is rejected at 5% Los(Level of significance)

4. A manfacturer claims that only 4% of his products supplied by him are defective. Sample of 600 products contained 36 defectives. Test the claim of the manufactrer.

Solution:

we set up H_0 : p = 0.04

H_1 : p > 0.04 (one tailed test)

Test Statistic $Z = \dfrac{p' - p}{\sqrt{\dfrac{pq}{n}}}$

Here $p = 0.04 \Rightarrow q = 1 - p = 0.96$

$p' = \dfrac{36}{600} = 0.06; \; n = 500$

$\therefore Z = \dfrac{0.06 - 0.04}{\sqrt{\dfrac{0.04 \times 0.96}{600}}} = 2.5$

Critical value :

The table value of Z = 1.645 at 5%L.o.s

Conclusion:

$|Z| = 2.5 > 1.645$, H_0 is rejected

$\therefore$ Manufacturer's claim is not acceptable

Type - IV: Test of significance for Difference of proportion of success in two samples:

To test the significance of the difference between the sample proportions p_1' and p_2'.

We formulate the null hypothesis H_0: $p_1 = p_2$

ie., the population proportions are equal

The alternative hypothesis is $H_1: p_1 \neq p_2$

The standard error of $p_1' - p_2' = \sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}$

Where $p = \dfrac{x_1 + x_2}{n_1 + n_2} = \dfrac{n_1 p_1' + n_2 p_2'}{n_1 + n_2}$

The test statistic is $Z = \dfrac{p_1' - p_2'}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} \sim N(0,1)$

Problems:

1. If a sample of 300 units of a manufactured product 65 units were found to be defective and in another sample of 200 units, there were 35 defectives. Is there significant difference in the proportion of defectives in the samples at 5% Los.

Solution:

$H_0: p_1 = p_2$ (ie., There is no significant difference in the proportion defectives in the samples)

The alternative hypothesis $H_1: p_1 \neq p_2$

Los: $\alpha = 0.05$

The test statistic is $Z = \dfrac{p_1' - p_2'}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

$p_1' = \dfrac{65}{300} = 0.22; \quad p_2' = 0.175$

$p = \dfrac{100}{500} = \dfrac{1}{5} \Rightarrow q = \dfrac{4}{5}$

$\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)} = \sqrt{\dfrac{4}{25}\left(\dfrac{1}{300} + \dfrac{1}{200}\right)} = 0.0365$

$$\therefore Z = \frac{0.22 - 0.175}{0.0365} = 1.233$$

Critical value :

The table value of Z at 5% Level = 1.96

Conclusion:

$|Z| < 1.96$, H_0 is accepted at 5% Los.

$\therefore$ The difference in the porportion of defectives in the samples is not significant

2. A machine puts out 16 imperfect articles in a sample of 500. After the machine is over-hauled in puts out 3 imperfect articles in a batch of 100. Has the machine improved?

Solution:

H_0: Machine has not been improved

ie., H_0 : $p_1 = p_2$

The alternative hypothesis H_1: $p_1 > p_2$ (one-tailed)

Los: $\alpha = 0.05$

The test statistic is $Z = \dfrac{p_1' - p_2'}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

Here $p_1' = \dfrac{16}{500} = 0.032$; $p_2' = 0.03$

$n_1 = 500$; $n_2 = 100$

$p = \dfrac{19}{600}$ and $q = \dfrac{581}{600}$

$\therefore Z = \dfrac{0.032 - 0.03}{\sqrt{\dfrac{19}{600} \times \dfrac{581}{600}\left(\dfrac{1}{500} + \dfrac{1}{100}\right)}} = 0.104$

$|Z| = 0.104$

Critical value :

The table value of Z for one tailed test Z = 1.645 at 5 Los

Conclusion:

$|Z| < 1.645$, H_0 is accepted at 5% Los.

The Machine has not improved due to overhaulding.

3. Before an increase in excise duty on tea, 800 perons out of a sample of 1000 persons were found to be tea drinkers. After an increse is excise duty. 800 people were tea drinkers in a sample of 1200 people. Test whether there is a significant decrease in the consumption of tea after the increase in excise duty at 5% Los

Solution:

H_0: the proportion of tea drinkers before and after the increase in excise duty are equal

ie., $p_1 = p_2$

H_1: $p_1 > p_2$

Los: $\alpha = 0.05$

The test statistic is $Z = \dfrac{p_1^{'} - p_2^{'}}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

Here $x_1 = 800$; $x_2 = 800$; $n_1 = 1000$;

$n_2 = 1200$; $p_1^{'} = \dfrac{800}{1000} = 0.8$; $p_2^{'} = \dfrac{800}{1200} = 0.67$

$p = \dfrac{x_1 + x_2}{n_1 + n_2} = \dfrac{1600}{2200} = \dfrac{8}{11} \Rightarrow q = \dfrac{3}{11}$

$\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)} = \sqrt{\dfrac{24}{121}\ 0.001 + 0.0008} = 0.0189$

$$\therefore Z = \frac{0.13}{0.0189} = 6.88 \Rightarrow |Z| = 6.88$$

Critical value: At 5% Los 1.645

Conclusion:

$$|Z| > 1.645, H_0 \text{ is rejected.}$$

$\therefore$ There is a significance decrease in the consumption of tea due to increase in excise duty.

Type - V: (Test of significance for the difference of S.D's of two large samples)

Let S_1 and S_2 be the S.D's of two indepedent samples of sizes n_1 and n_2 respectively

The null hypothesis $H_0: \mu_1 = \mu_2$;

ie., the sample S.D's do not differ significantly.

The Alternative Hypothesis $H_1: \mu_1 \neq \mu_2$

the test statistic is $Z = \dfrac{S_1 - S_2}{S.E(S_1 - S_2)} \sim N(0,1)$ for large 'n'

ie., If σ_1 and σ_2 are known,

$$Z = \frac{S_1 - S_2}{\sqrt{\dfrac{\sigma_1^2}{2n_1} + \dfrac{\sigma_2^2}{2n_2}}} \sim N(0,1)$$

(or)If σ_1 and σ_2 are not known,

$$Z = \frac{S_1 - S_2}{\sqrt{\dfrac{S_1^2}{2n_1} + \dfrac{S_2^2}{2n_2}}}$$

If $|Z| > Z_\alpha$, H_0 is rejected at $\alpha\%$ level, otherwise H_0 is accepted

Problems:

1. The sample of sizes 1000 and 800 gave the following results

	Mean	S.D
Sample I	17.5	2.5
Sample II	18	2.7

Assuming that the samples are indepedent, test whether the two samples may be regarded as drawn from the universe with same S.D's at 1% Level.

Solution:

We set up $H_0 : \sigma_1 = \sigma_2$;

ie., two samples maybe regarded as drawn from the universe with same S.D's

$H_1 : \sigma_1 \neq \sigma_2$

Test statistic $Z = \dfrac{S_1 - S_2}{\sqrt{\dfrac{S_1^2}{2n_1} + \dfrac{S_2^2}{2n_2}}}$

Here $n_1 = 1000$; $n_2 = 800$; $S_1 = 2.5$; $S_2 = 2.7$

$\therefore Z = \dfrac{2.5 - 2.7}{\sqrt{\dfrac{(2.5)^2}{2000} + \dfrac{(2.7)^2}{1600}}} = \dfrac{-0.2}{\sqrt{0.3125 + 0.455625}}$

$\Rightarrow |Z| = 2.282$

Critical value :

At 1% Los, the tabulated value is 2.58

Conclusion:

Since $|Z| < 2.58$, H_0 is accepted at 1% Los.

$\therefore$ The two samples may be regarded as drawn from the universe with the same S.D's

2. In a survey of incomes of two classes of workers, two random samples gave the following results. Examine whether the differences between (i) the means and (ii) the S.D's are significant.

Sample	Size	Mean annualincome (Rs)	S.D in Rs
I	100	582	24

| | II | 100 | 546 | 28 |

Examine also whether the samples have been drawn from a population with same S.D

Solution:

(i) We set up $H_0 : \mu_1 = \mu_2$;

ie., the difference is not significant

$H_1 : \mu_1 \neq \mu_2$

Here it is two tailed test

Test statistic $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}} = \dfrac{582 - 546}{\sqrt{\dfrac{(24)^2}{100} + \dfrac{(28)^2}{100}}}$

$\therefore \ Z = \dfrac{360}{\sqrt{(24)^2 + (28)^2}} = 9.76$

$\Rightarrow \ |Z| = 9.76$

Critical value :

At 5% Los, the table value of Z is 1.96

Conclusion:

Since $|Z| > 1.96$, H_0 is rejected at 5% Los.

$\therefore$ There is a significant difference in the means in the two samples.

(ii) $H_0 : \sigma_1 = \sigma_2$
$H_1 : \sigma_1 \neq \sigma_2$
Here it is two tailed test
Los: $\alpha = 0.05$

Test statistic $Z = \dfrac{S_1 - S_2}{\sqrt{\dfrac{S_1^2}{2n_1} + \dfrac{S_2^2}{2n_2}}} = \dfrac{24 - 28}{\sqrt{\dfrac{(24)^2}{200} + \dfrac{(28)^2}{200}}}$

$\therefore \quad Z = \dfrac{-40}{\sqrt{288+392}} = -1.53$

$\Rightarrow \quad |Z| = 1.53$

Critical value :

At 5% Los, the table value of Z is 1.96

Conclusion:

Since $|Z| < 1.96$, H_0 is accepted at 5% Los.

$\therefore$ The difference between the sample S.D's is not significant.

Hence we conclude that the two samples have been drawn from population with the same S.D's

3. Two machines A and B produced 200 and 250 items on the average per day with a S.D of 20 and 25 items reply on the basis of records of 50 day's production. Can you regard both machine's equally efficient at 1% Los.

Solution:

(i) $H_0 : \sigma_1 = \sigma_2$; ie., the two machines aer equally efficient

$H_1 : \sigma_1 \neq \sigma_2$

Los: $\alpha = 0.05$

Test statistic $Z = \dfrac{S_1 - S_2}{\sqrt{\dfrac{S_1^{\,2}}{n_1} + \dfrac{S_2^{\,2}}{n_2}}}$

$n_1 = 200 \times 50; \quad S_2 = 25$

$n_2 = 250 \times 50; \quad S_1 = 20$

$\therefore \quad Z = \dfrac{(20-25) \times \sqrt{50}}{\sqrt{\dfrac{400}{400} + \dfrac{625}{500}}} = \dfrac{-5\sqrt{50}}{\sqrt{1+1.25}} = -23.57$

$\Rightarrow \quad |Z| = 23.57$

Critical value :

At 1% Los, the table value of Z is 2.58

Conclusion:

Since $|Z| > 2.58$, H_0 is rejected at 1% Los.

We conclude that the both machines are not equally efficient at 1% Los

Small sample Tests (t - Test):

Definition:

Consider a random sample $\{x_1, x_2, \ldots x_n\}$ of size 'n' drawn from a Normal population with mean μ and variance σ^2.

$$\text{Sample mean } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The unbiased estimate of the pop.variance σ^2 is denoted as s^2.

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

The student's t-statistic is defined as t = $\dfrac{|\bar{x} - \mu|}{s}\sqrt{n}$, Where n = sample size

The degree of freedon of this statistic

$$V = n - 1$$

Type I:

To test the significance of a single mean (For small samples)

$$\text{Test Statistic t} = \frac{\bar{x} - \mu}{\dfrac{S.D}{\sqrt{n-1}}} = \frac{\bar{x} - \mu}{\dfrac{S}{\sqrt{n}}}$$

s = sample S.D and

$$S^2 = \frac{1}{n-1}\sum(x - \bar{x})^2 \text{ (or) } S = \sqrt{\frac{ns^2}{n-1}}$$

If the computed value of t is greater than the critical value t_α, H_o is rejected

(or) if $|t| < t_\alpha$, the null hypothesis H_o is accepted at α level.

1. A machinist is making engine parts with axle diameter of 0.700 inch. A random sample of 10 parts shows a mean diamter of 0.742 inch with a S.D of 0.40. Test whether the work is meeting the specification at 5% Los

Solution:

Given that n =10; $\bar{x} = 0.742$ inches

$\mu = 0.700$ inches $S = \sqrt{\dfrac{ns^2}{n-1}} = \sqrt{\dfrac{10 \times (0.40)^2}{9}} = 0.4216$

s = 0.40 inches   S = 0.42

Null hypothesis H_0 : the product is confirming to specification ie., there is no significant difference between $\bar{x}$ and μ

H_0 : $\mu = 0.700$ inches

H_1 : $\mu \neq 0.700$ inches

Test Statistic $t = \dfrac{|\bar{x} - \mu|}{s}\sqrt{n} = 0.316$

degrees of freedom = n-1 = 9

Table value of t at 5% level = 2.26

$\therefore$ the product is meeting the specification.

2. Ten individuals are chosen at random from a population and their heights are found to be in inches 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of this data, discuss the suggestion that the mean height in the universe is 66 inches.

Solution:

x :	63	63	66	67	68	69	70	70	71	71
$(x-\bar{x})^2$:	23.04	23.04	3.24	0.64	0.04	1.44	4.84	4.84	10.24	10.24

$\therefore \sum x = 678$ and $\sum (x - \bar{x})^2 = 81.6$

$$\bar{x} = \frac{\sum x}{n} = \frac{678}{10} = 67.8$$

$$S = \sqrt{\frac{\sum (x - \bar{x})^2}{9}} = \sqrt{\frac{81.6}{9}} = 3.011$$

Let $H_0 : \mu = 66$ the mean and height if the universe is 66 inches

and $H_1 : \mu \neq 66$

Los $\alpha = 0.05$

Test Statistic $t = \frac{|\bar{x} - \mu|}{s}\sqrt{n} = \frac{67.8 - 66}{3.011}\sqrt{10} = 1.89$

Table value of t for 9 d.f at 5% Los is $t_0 = 2.2$

Since $|t| < t_0, H_0$ is accepted at 5% level.

$\therefore$ The mean height of universe of 66 is accepted.

Type II: (Test of significance of difference of mean)

Test Statistic $t = \dfrac{\bar{x_1} - \bar{x_2}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

Where $S^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$ (or)

$$S^2 = \frac{\sum(x_1 - \bar{x_1})^2 + \sum(x_2 - \bar{x_2})^2}{n_1 + n_2 - 2}$$

The number of degrees of freedom $= V = n_1 + n_2 - 2$

The calculated value of t is less than the table value of t for d.f $= n_1 + n_2 - 2$, H_0 is accepted

Otherwise H_0 is rejected at the selected Los

1. Two independent samples from normal pop's with equal variances gave the following

results

Sample	Size	Mean	S.D

| 1 | 16 | 23.4 | 2.5 |
| 2 | 12 | 24.9 | 2.8 |

Test for the equations of means.

Solution:

(i) We set up $H_0 : \mu_1 = \mu_2$; ie., there is no significant difference between their means

$H_1 : \mu_1 \neq \mu_2$

Los: $\alpha = 0.05$

Test Statistic $t = \dfrac{\overline{x_1} - \overline{x_2}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ Where $S^2 = \dfrac{n_1 s_1{}^2 + n_2 s_2{}^2}{n_1 + n_2 - 2}$

Given $\overline{x_1} = 23.4; \quad n_1 = 16; \quad s_1 = 2.5$

$\overline{x_2} = 24.9; \quad n_2 = 12; \quad s_2 = 2.8$

$S^2 = \dfrac{n_1 s_1{}^2 + n_2 s_2{}^2}{n_1 + n_2 - 2} = \dfrac{16(2.5)^2 + 12(2.8)^2}{16 + 12 - 2}$

$= \dfrac{100 + 94.08}{26} = 7.465$

$S = 2.732$

$\therefore t = \dfrac{23.4 - 24.9}{2.732\sqrt{\dfrac{1}{16} + \dfrac{1}{12}}} = -1.438$

$\therefore |t| = 1.438$

Number of degrees of freedom $= n_1 + n_2 - 2 = 26$

Critical value :

The table value of t for 26 d.f at 5% Los is

$t_{0.05} = 2.056$

Conclusion:

Since the calculated value of t is less than table value of t,

H_0 is accepted at 5% Los.

$\therefore$ There is no significant difference between their means

2. Two independent samples of 8 and 7 items respectively had the following values

Sample I :   9   13   11   11   15   9   12   14

Sample II : 10   12   10   14   9   8   10

Is the difference between the means of the samples significant?

Solution:

We set up $H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

Hence it is a two tailed test

Los: $\alpha = 0.05$

Test Statistic $t = \dfrac{\overline{x_1} - \overline{x_2}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

Where $S^2 = \dfrac{\sum(x_1 - \overline{x_1})^2 + \sum(x_2 - \overline{x_2})^2}{n_1 + n_2 - 2}$

Computation of t:

x_1	$d_1 = (x_1 - \overline{x_1})$ $d_1 = x_1 - 11.75$	$d_1^2 = (x_1 - \overline{x_1})^2$	x_2	$d_2 = (x_2 - \overline{x_2})$ $d_2 = x_2 - 10.43$	$d_2^2 = (x_2 - \overline{x_2})^2$
9	-2.75	7.5625	10	-0.43	0.1849
13	1.25	1.5625	12	1.57	2.4649
11	-0.75	0.5625	10	-0.43	0.1849
11	-0.75	0.5625	14	3.57	12.7449
15	3.25	10.5625	9	-1.43	2.0449
9	-2.75	7.5625	8	-2.43	5.9049
12	0.25	0.0625	10	-0.43	0.1849
14	2.25	5.0625			
	$\sum d_1 = 3.5$	$\sum d_1^2 = 33.5$		$\sum d_2 = -0.01$	$\sum d_2^2 = 23.7143$

$$\overline{x_1} = 11 + \frac{6}{8} = 11.75$$

$$\overline{x_2} = 10 + \frac{3}{7} = 10.43$$

$$\sum (x_1 - \overline{x_1})^2 = \sum d_1^2 - \frac{\left(\sum d_1\right)^2}{n_1} = 38 - \frac{36}{8} = 33.5$$

$$\sum (x_2 - \overline{x_2})^2 = \sum d_2^2 - \frac{\left(\sum d_2\right)^2}{n_2} = 25 - \frac{9}{7} = 33.5$$

$$\therefore S^2 = \frac{33.5 + 23.71}{8 + 7 - 2} \implies S = 2.097$$

$$\therefore t = \frac{\overline{x_1} - \overline{x_2}}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$= \frac{11.75 - 10.43}{2.097\sqrt{\dfrac{1}{8} + \dfrac{1}{7}}}$$

$$t = 1.218$$

$$\text{d.f} = 8 + 7 - 2 = 13$$

Critical value:

The table value of t for 13 d.f at 5% level is 2.16

Conclusion:

Since $|t| < 2.16$, H_0 is accepted

$\therefore$ There is no significant difference between the means of the two samples.

Type III:

Testing of significance of the difference in means paired data.

When the two samples are of the same sizes and the data are paired

the test statistic is $t = \dfrac{\bar{d}}{\dfrac{S}{\sqrt{n}}}$

Where $\bar{d} = $ mean of differences

and $S = \sqrt{\dfrac{\sum (d - \bar{d})^2}{n - 1}}$

Degrees of freedom = n-1

1. Elevan school boys were given a test in painting. They were given a month's further tution and a second test of equal difficulty was held at the end of the month. Do the marks give evidence that the students have beneifit by extra coaching?

Boys:	1	2	3	4	5	6	7	8	9	10	11
First Test (marks)	25	23	19	22	21	19	22	21	25	18	20
Second test (marks)	26	22	22	19	23	21	24	24	25	22	18

Solution:

$H_0 : \mu =$ the student have not been benefited by extra coaching.

ie., The mean of the difference between the marks of the two tests is zero

ie., $H_0 : \bar{d} = 0$

$H_1 : \bar{d} > 0$

Los: $\alpha = 0.05$ (or) 5%

the test statistic is t = $\dfrac{\bar{d}}{\dfrac{S}{\sqrt{n}}}$

S. No	1	2	3	4	5	6	7	8	9	10	11
d = x - y	-1	1	-3	3	-2	-2	-2	-3	0	-4	2
$d - \bar{d}$	0	2	-2	4	-1	-1	-1	-2	1	-3	3
$d - \bar{d}^{\,2}$	0	4	4	16	1	1	1	4	1	9	9

$\sum d = -11; \quad \bar{d} = \dfrac{\sum d}{n} = \dfrac{-11}{11} = -1$

$\sum d - \bar{d}^{\,2} = 50$

$S = \sqrt{\dfrac{\sum (d - \bar{d})^2}{n-1}} = \sqrt{\dfrac{50}{10}} = \sqrt{5} = 2.236$

$$\therefore t = \frac{\overline{d}}{\frac{S}{\sqrt{n}}} = \frac{-1}{\frac{2.236}{\sqrt{11}}}$$

$$\therefore |t| = \frac{1}{0.625} = 1.48$$

No. of d.f = 11-1 = 10

Critical value:

At 5% Los, the table value of t at 10 degree freedom is 1.812

Conclusion:

$|t| < 1.812$, H_0 is accepted at 5% Los.

$\therefore$ The students have not been benefited by extra-coaching.

2. The scores of 10 candidates prior and after training are given below,

Prior : 84   48   36   37   54   69   83   96   90   65

After : 90   58   56   49   62   81   84   86   84   75

Is the training effective?

Solution:

We set up H_0 : the training is not effective

ie., H_0 : $\overline{d} = 0$

H_1: $\overline{d} > 0$

the test statistic is $t = \frac{\overline{d}}{\frac{S}{\sqrt{n}}}$

S. No	1	2	3	4	5	6	7	8	9	10
d = x - y	-6	-10	-20	-12	-8	-12	-1	10	6	-10
d - $\overline{d}$	0.3	-3.7	-13.7	-5.7	-1.7	-5.7	5.3	16.3	12.3	-3.7
d - $\overline{d}$ ^{2}	0.09	13.69	187.69	32.49	2.89	32.49	28.09	265.69	151.29	13.69

$$\sum d = -63; \quad \bar{d} = \frac{\sum d}{n} = \frac{-63}{10} = -6.3$$

$$\sum \left(d - \bar{d} \right)^2 = 728.1$$

$$S = \sqrt{\frac{\sum (d - \bar{d})^2}{n-1}} = \sqrt{\frac{728.1}{9}} = \sqrt{80.9} = 8.994$$

$$S = 8.994$$

$$\therefore t = \frac{-6.3}{\dfrac{8.994}{\sqrt{10}}} = \frac{-6.3}{2.844} = -2.21$$

$$|t| = 2.21$$

Degrees of freedom V= n-1 = 10-1 = 9

Critical value:

At 5% Los, the table value of t at 9 degree freedom is 2.262

Conclusion:

$|t| < 2.262$, H_0 is accepted at 5% Los.

$\therefore$ There is no effective in the training.

Variance Ratio Test (or) F-test for equality of variances

This test is used to test the significance of two or more sample estimates of population variance

The F-statistic is defined as a ratio of unbiased estimate of population variance

$$F = \frac{S_1^2}{S_2^2}; \quad \text{Where } S_1^2 = \frac{\sum \left(x_1 - \bar{x}_1 \right)^2}{n_1 - 1}$$

$$S_2^2 = \frac{\sum \left(x_2 - \bar{x}_2 \right)^2}{n_2 - 1}$$

$\therefore$ The distribution of $F = \dfrac{S_1^2}{S_2^2}$ $S_1^2 > S_2^2$ is given by the following p.d.f

If S_1^2 and S_2^2 are the variances of two sample of sizes n_1 and n_2 respectively, the estimate

of the population variances based on these samples are respectively

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1}; \quad S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

d.f $V_1 = n_1 - 1 \ \& \ V_2 = n_2 - 1$

While defining the statistic F, the large oftwo variances is always placed in the numerator and

smaller in the denominator

Test of significance for equality of population variances

Consider two independent R, samples $x_1, x_2, \ldots \ldots x_{n_1}$ & $y_1, y_2, \ldots \ldots y_{n_2}$ from normal populations

The hypothesis to be tested is

"The population variances are same".

we set up: $H_0 : \sigma_1^2 = \sigma_2^2$

& $H_1 : \sigma_1^2 \neq \sigma_2^2$

The test statistic $F = \dfrac{S_1^2}{S_2^2}$ $S_1^2 > S_2^2$

$$S_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n} \left(x_i - \bar{x} \right)^2 \text{ and } S_1^2 = \frac{1}{n_1 - 1} \sum_{j=1}^{n} \left(y_j - \bar{y} \right)^2$$

F distribution with d.f $V_1 = n_1 - 1 \ \& \ V_2 = n_2 - 1$

Problems:

1. It is known that the mean diameters o rivets produced by two firms A and B are practically the same but the standard deviations may differ.

   For 22 rivets produced by A, the S.D is 2.9 m, while for 16 rivets manufactured by B, the S.D is 3.8 m. Test whether the products of A have the same variabilty as those of B

   Solution:

   $H_0 : \sigma_1^2 = \sigma_2^2$

ie., variability for the two types of products are same.

Los: $\alpha = 0.05$ (or) 5%

The test statistic $\quad F = \dfrac{S_1^{\ 2}}{S_2^{\ 2}} \quad S_1^{\ 2} > S_2^{\ 2}$

Given, $n_1 = 22; \quad n_2 = 16$

$S_1 = 2.9; \quad S_2 = 3.8$

$S_1^{\ 2} = \dfrac{n_1 s_1^{\ 2}}{n_1 - 1} = \dfrac{22(2.9)^2}{22 - 1} = 8.81$

$S_2^{\ 2} = \dfrac{n_2 s_2^{\ 2}}{n_2 - 1} = \dfrac{16(3.8)^2}{16 - 1} = 15.40$

$F = \dfrac{S_2^{\ 2}}{S_1^{\ 2}} \quad S_2^{\ 2} > S_1^{\ 2}$

$\quad = \dfrac{15.40}{8.81}$

$F = 1.748$

Number of degrees of freedom are $V_1 = 16 - 1 = 15$

$\hspace{5cm} V_2 = 22 - 1 = 21$

Critical value:

At 5% Los, the table value of F at d.f $(15, 21)$ is $F = 2.18$

Conclusion:

$F < 2.18$, H_0 is accepted at 5% Los.

$\therefore$ Variability for two types of products may be same.

2. Two random samples of sizes 8 and 11, drawn from two normal populations are characterized as follows

	Size	Sum of observations	Sum of square of observations
Sample I	8	9.6	61.52
Sample II	11	16.5	73.26

You are to decide if the two populations can be taken to have the same variance.

Solution:

Let x and y be the observations of two samples

we set up: $\quad H_0 : \sigma_1^{\ 2} = \sigma_2^{\ 2}$

& $H_1: \sigma_1^2 \neq \sigma_2^2$

For sample I

$$s_1^2 = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$= \frac{61.52}{8} - \left(\frac{9.6}{8}\right)^2$$

$$= 7.69 - (1.2)^2 = 7.69 - 1.44$$

$$s_1^2 = 6.25$$

For sample II

$$s_2^2 = \frac{\sum y^2}{n} - \left(\frac{\sum y}{n}\right)^2$$

$$= \frac{73.26}{11} - \left(\frac{16.5}{11}\right)^2$$

$$= 6.66 - (1.5)^2 = 6.66 - 2.25$$

$$s_2^2 = 4.41$$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{8(6.25)}{7} = 7.143$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{11(4.41)}{10} = 4.851$$

$$F = \frac{S_2^2}{S_1^2} \qquad S_2^2 > S_1^2$$

$$= \frac{7.143}{4.851} = 1.472$$

$$F = 1.472$$

Number of degrees of freedom are $V_1 = n_1 - 1 = 8 - 1 = 7$

$$V_2 = n_2 - 1 = 11 - 1 = 10$$

Critical value:

The table value of F for (7,10) d.f at 5% Los is 3.14

Conclusion:

Since $|F| < 3.14$, H_0 is accepted at 5% level

∴ Variances of two populations may be same.

Variability for two types of products may be same.

Chi-Square Test

Definition

If O_i (i = 1, 2, ... n) are set of observed (experimental) frequencies and $E_i (i = 1, 2, ... n)$

are the corresponding set of expected frequncies, then the statistic

χ^2 is defined as

$$\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i}{E_i}^2$$

The degree of freedom is $v = n - 1$

For fitting Binomial distribution $v = n - 1$

For fitting Poisson distribution $v = n - 2$

For fitting Normal distribution $v = n - 3$

Chi-square Test of Goodness of fit

If the calculated value of χ^2 is less than the table value at a specified Los.

The fit is considered to be good

otherwise the fit is considered to be poor.

Conditions for applying χ^2 Test

For the validity of chi-square test of "goodness of fit" between theory and experiment following

Conditions must be satisfied.

(i)   The sample of observations should be independent
(ii)   Constraints on the cell frequncies. If any, should be linear.
(iii)   N, the total frequency should be reasonably large, say greater than 50.
(iv) N_0 theoretical cell frequency should be less than 5, If any theoretical cell

frequency less than 5, the for application χ^2 test
It is pooled with the preceeding or succeeding frequency so that the pooled
frequency is greater than 5 and finally adjust for the d.f lost in pooling.

Problems

1. The following table gives the number of aircraft accident that occured during the various days

of the week. Test whether the acidents are uniformly distributed over the week.

Days   : Mon   Tue   Wed   Thu   Fri   Sat   Total
No.of accidents :   14   18   12   11   15   14   84

Solution:

We set up H_0 : The accidents are uniformly distributed over the week

Los $\alpha = 0.05$

Test Statistic $\chi^2 = \sum_{i=1}^{n} \dfrac{O_i - E_i}{E_i}^2$

Under the nul hypothesis,

The expected frequency of the on each day $= \dfrac{84}{6} = 14$

O_i :   14   18   12   11   15   14

E_i :   14   14   14   14   14   14

$\chi^2 = \dfrac{14-14}{14}^2 + \dfrac{18-14}{14}^2 + \dfrac{12-14}{14}^2 + \dfrac{11-14}{14}^2 + \dfrac{15-14}{14}^2 + \dfrac{14-14}{14}^2$

$= 1.143 + 0.286 + 0.643 + 0.071$

= 2.143

Number of degrees of freedom $V = n - 1 = 7 - 1 = 6$

Critical value:

The tablulated value of χ^2 at 5% for 6 d.f is 12.59

Conclusion:

Since $\chi^2 < 12.59$, we accept the null hypothesis

$\therefore$ We conclude that the accidents are uniformly distributed over the week.

2. The theory predicts the population of beans in the four groups A, B , C and D should be 9:3:3:1. In an experiment among 1600 beans, the number in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory?

Solution:

We set up the null hypothesis

H_0 : The theory fits well into the experiment

ie., the experimental results supports the theory

Total Number of beans = 1600

Divide these beans in the ratio 9:3:3:1

To calculate the expected frequencies

$E(882) = \dfrac{9}{16} \times 1600 = 900$

$E(313) = \dfrac{3}{16} \times 1600 = 300$

$E(287) = \dfrac{3}{16} \times 1600 = 300$

$E(118) = \dfrac{1}{16} \times 1600 = 100$

O_i :   882   313   287   118

E_i : 900   300   300   100

Test Statistic $\chi^2 = \sum\limits_{i=1}^{n} \dfrac{\left(O_i - E_i\right)^2}{E_i}$

$$\chi^2 = \frac{\left(882 - 900\right)^2}{900} + \frac{\left(313 - 300\right)^2}{300} + \frac{\left(287 - 300\right)^2}{300} + \frac{\left(118 - 100\right)^2}{100}$$

$$= 0.36 + 0.563 + 0.563 + 3.24$$

$$\chi^2 = 4.726$$

Critical Value:

The table value of χ^2 at 5% for 3 d.f is 7.815

Conclusion:

Since $\chi^2 < 7.815$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that there is a very good correspondent between theory and experiment

3. 4 coins were tossed 160 times and the following results were obtained.

No. of heads   :   0   1   2   3   4

Frequency   :   19   50   52   30   9   160

0   50   104   90   36   280

Test the goodness of fit with the help of χ^2 on the assumption that the coins are unbiased

Solution:

We set up, the null hypothesis, the coins are unbiased:

The probability if getting the success of heads   is p = $\dfrac{1}{2}$

q = 1- p = $\dfrac{1}{2}$

When 4 coins are tossed, the probability of getting 'r' heads is given by,

$P(x = r) = n_{C_r} \ p^r q^{n-r} \ ; \ r = 0, 1, 2, 3, 4$

$$= 4_{C_r} \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{4-r}$$

$$= 4_{C_r} \left(\frac{1}{2}\right)^4$$

$\therefore \ P(x = r) = 4_{C_r} \ \dfrac{1}{16} \quad r = 0, 1, 2, 3, 4$

The expected frequencies of getting 0, 1, 2, 3, 4 heads are given by $1604_{C_r} \ \dfrac{1}{16}$

$$= 104_{C_r}, \ r = 0, 1, 2, 3, 4$$

$$= 10, 40, 60, 40, 10$$

O_i	:	19	50	52	30	9
E_i	:	10	40	60	40	10
		26	48	43	26	12

Test Statistic $\chi^2 = \displaystyle\sum_{i=1}^{n} \dfrac{\left(O_i - E_i\right)^2}{E_i}$

$$\chi^2 = \frac{(19-10)^2}{10} + \frac{(50-40)^2}{40} + \frac{(52-60)^2}{60} + \frac{(30-40)^2}{40} + \frac{(9-10)^2}{40}$$

$$= 8.1 + 2.5 + 1.067 + 2.5 + 0.1$$

$\chi^2 = 14.267$

D.f $V = n-1 = 5-1 = 4$

Critical value:

The table value of χ^2 for 4 d.f at 5% Los is 9.488

Conclusion:

Since $\chi^2 > 9.488$, H_0 is rejected at 5% Los

∴ The coins are biased

4. The follwoing table shows the distribution of goals in a football match

No. of goals   :   0   1   2   3   4   5   6   7

No. of mistakes :   95   158   108   63   40   9   5   2

Fit a poisson distribution and test the goodness of fit.

Solution:

Fitting of poisson distribution

x :   0   1   2   3   4   5   6   7

f :   95   158   108   63   40   9   5   2

$\sum fx = 812$ and $\sum f = 480$

$\therefore \ \bar{x} = \lambda = \dfrac{\sum fx}{\sum f} = \dfrac{812}{480} = 1.7$

∴ The expected frequencies are computed by

$= 480 \times \dfrac{e^{-1.7}(1.7)^r}{r!} \qquad r = 0, 1, 2, 3, 4, 5, 6, 7$

$= 88, 150, 126, 72, 30, 10, 3, 1$

We set up H_0 : The fit is good

Test Statistic $\chi^2 = \sum\limits_{i=1}^{n} \dfrac{{O_i - E_i}^2}{E_i}$

O_i :   95   158   108   63   40 $\underbrace{9 \qquad 5 \qquad 2}_{16}$

E_i :   88   150   126   72   30 $\underbrace{10 \qquad 3 \qquad 1}_{14}$

$\chi^2 = \dfrac{{O-E}^2}{E} = \dfrac{{95-88}^2}{88} + \dfrac{{158-150}^2}{150} + \dfrac{{108-126}^2}{126} + \dfrac{{40-30}^2}{30} + \dfrac{{16-14}^2}{14} + \dfrac{{63-72}^2}{72}$

$= 0.56 + 0.43 + 2.57 + 3.33 + 1.12 + 0.29$

$\chi^2 = 8.30$

Number of degrees of freedom $V = n - 2 = 6 - 2 = 4$

Critical value:

   The table value of χ^2 at 5% Los for 4 d.f is 9.483

Conclusion:

   Since $\chi^2 < 9.483$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

5. Apply the χ^2 test of goodness of fit to the follwoing data

O_i	:	1	5	20	28	42	22	15	5	2
E_i	:	1	6	18	25	40	25	18	6	1

Solution:

   H_0 : The fit is good

   $\alpha = 0.05$ (or) 5%

O_i	:	$\underbrace{1 \quad 5}_{6}$	20	28	42	22	15	$\underbrace{5 \quad 2}_{7}$
E_i	:	$\underbrace{1 \quad 6}_{7}$	18	25	40	25	18	$\underbrace{6 \quad 1}_{7}$

$n = 7$

Test Statistic $\chi^2 = \sum\limits_{i=1}^{n} \dfrac{\left(O_i - E_i\right)^2}{E_i}$

$\chi^2 = \dfrac{(6-7)^2}{7} + \dfrac{(20-18)^2}{18} + \dfrac{(28-25)^2}{25} + \dfrac{(42-40)^2}{40} + \dfrac{(22-25)^2}{25} + \dfrac{(15-18)^2}{18} + \dfrac{(7-7)^2}{7}$

   $= 0.143 + 0.222 + 0.36 + 0.1 + 0.36 + 0.5 + 0$

$\chi^2 = 1.685$

d.f $V = n - 1 = 7 - 1 = 6$

Critical value:

At 5% Los, the table value of χ^2 for 6 d.f is 12.592

Conclusion:

Since $\chi^2 < 12.592$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

6. The follwoing table shows the number of electricity failures in a town for a period of 180 days

Failures	:	0	1	2	3	4	5	6	7
No. of days	:	12	39	47	40	20	17	3	2

Use χ^2, examine whether the data are poisson distributed.

Solution:

Fitting of poisson distribution

x :	0	1	2	3	4	5	6	7
f :	12	39	47	40	20	17	3	2
fx :	0	39	94	120	80	85	18	14

$\sum fx = 450$ and $\sum f = 180$

$\therefore \; \bar{x} = \lambda = \dfrac{\sum fx}{\sum f} = \dfrac{450}{180} = 2.5$

$\therefore$ The expected frequencies are computed by

$= 180 \times \dfrac{e^{-2.5}(2.5)^r}{r!}$ $r = 0, 1, 2, 3, 4, 5, 6, 7$

$E_i = 15, 37, 46, 38, 24, 12, 5, 2$

r = 0 1 2 3 4 5 6 7

We set up H_0 : The fit is good

O$_i$: 12   39   47   40   20   17   3 ⌣ 2
                                                    5

E_i : 15   37   46   38   24   12 $\underbrace{5 \quad 2}_{7}$

$\therefore$ Test Statistic χ^2

$$\chi^2 = \frac{12-15^{\,2}}{15} + \frac{39-37^{\,2}}{37} + \frac{47-46^{\,2}}{46} + \frac{40-38^{\,2}}{38} + \frac{20-24^{\,2}}{24} + \frac{17-12^{\,2}}{12} + \frac{5-7^{\,2}}{7}$$

$= 0.6 + 0.108 + 0.022 + 0.105 + 0.667 + 2.083 + 0.5 + 1$

$\chi^2 = 4.156$

d.f V = n-1 = 7 - 1 = 6

Critical value:

At 5% Los, the table value of χ^2 for 6 d.f is 12.592

Conclusion:

Since $\chi^2 < 12.592$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

Test for Independence of Attributes

Attribute A

		A_1	A_2..............A_j.................	A_t	Total
Attribute	B_1	O_{11}	O_{12}...............O_{1j}.................	O_{1t}	(B_1)
B	B_2				
	.				
	.				
	B_i	O_{i1}	O_{i2}...............O_{ij}.................	O_{it}	(B_i)
	.	.			
	.	.			

	B_s	O_{s1}	O_{s2}...............O_{sj}.................. O_{st}		(B_s)
	Total	(A_1)	(A_2)...............(A_i).................(A_t)		N

Attribute A

		A_1	A_2...............A_j................. A_t	Total
Attribute	B_1	O_{11}	O_{12}...............O_{1j}................. O_{1t}	(B_1)
B	B_2			
	.			
	.			
	.			
	B_i	O_{i1}	O_{i2}...............O_{ij}................. O_{it}	(B_i)
	.	.		
	.	.		
	.	.		
	B_s	O_{s1}	O_{s2}...............O_{sj}................. O_{st}	(B_s)
	Total	(A_1)	(A_2)...............(A_i).................(A_t)	N

Such a table is called (s × t) consistency table

Here,   N → Total Frequency

O_{ij} → Observed frequency of $(i, j)^{th}$ cell

The expected frequency e_{ij} obtained by the rule

$$e_{ij} = \frac{\text{row total } B_i \quad \text{Column total } A_j}{N}$$ Where i = 1, 2, 3 .... s

j = 1, 2 .... t

Degrees of freedom associated with s × t consistency table = (s - 1) ×(t - 1)

Chi-square table for 2×2 consistency table

In a 2×2 consistency table where in the frequencies are $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,   the value of χ^2 is

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(a+c)(c+d)(b+d)}$$

Problems :

1. An opinion poll was conducted to find the reaction to a proposed civic reform in 100 members

of each of the two political parties the information is tabulated below

	Favorable	Unfavorable	Indifferent
Party A	40	30	30
Party B	42	28	30

Test for Indepedence of reduction with the party affiliations.

Solution:

We set up H_0 : Reactions and party affiliations are independent

The expected frequencies are calculated by

$$
\begin{array}{cccc}
 & & & \text{Total} \\
40 & 30 & 30 & 100 \\
42 & 28 & 30 & 100 \\
\text{Total } 82 & 58 & 60 & 200
\end{array}
$$

	Favorable	Unfavorable	Indifferent
Party A	$\dfrac{82 \times 100}{200} = 41$	$\dfrac{58 \times 100}{200} = 29$	$\dfrac{60 \times 100}{200} = 30$
Party B	$\dfrac{82 \times 100}{200} = 41$	$\dfrac{58 \times 100}{200} = 29$	$\dfrac{60 \times 100}{200} = 30$

$\therefore$ Test Statistic χ^2

$$\chi^2 = \sum_{i=1}^{n} \frac{\overline{O_i - E_i}^2}{E_i}$$

O_i : 40 30 30 42 28 30

E_i : 41 29 30 41 29 30

$$\chi^2 = \frac{(40-41)^2}{41} + \frac{(30-29)^2}{29} + \frac{(30-30)^2}{30} + \frac{(42-41)^2}{41} + \frac{(28-29)^2}{29} + \frac{(30-30)^2}{30}$$

$$= 0.024 + 0.024 + 0.034 + 0.034$$

$$\chi^2 = 0.116$$

Number of degrees of freedom $= (2-1)(3-1) = 2$

Critical value:

At 5% Los, the table value of χ^2 for 2 d.f is 5.99

Conclusion:

Since $\chi^2 < 5.99$, H_0 is accepted at 5% Los.

$\therefore$ The independence of reactions with the party affiliations may be correct.

2. In a locality 100 persons were randomly selected and asked about their educational

achievements. The results are given below.

	Education		
	Middle	**High School**	**College**
Male	**10**	**15**	**25**
Female	**25**	**10**	**15**

Can you say that education depends on sex?

3. The following table gives the classification of 100 workers according to sex and the nature

of work. Test whether nature of work is independent of the sec of the worker.

		Skilled	Unskilled	Total
	Male	40	20	60
Sex	Female	10	30	40
	Total	50	50	

Solution:

H_0 : Nature of work is independent of the sex of the worker

Under H_0, the expected frequencies are

$$E(40) = \frac{60 \times 50}{100} = 30; \qquad E(20) = \frac{60 \times 50}{100} = 30$$

$$E(10) = \frac{40 \times 50}{100} = 20; \qquad E(30) = \frac{40 \times 50}{100} = 20$$

$\therefore$ Test Statistic χ^2

$$\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i^{\ 2}}{E_i}$$

O_i : 40   20   10   30

E_i : 30   30   20   20

$$\chi^2 = \frac{40-30^{\ 2}}{30} + \frac{20-30^{\ 2}}{30} + \frac{10-20^{\ 2}}{20} + \frac{30-20^{\ 2}}{20}$$

$$= 3.333 + 3.333 + 5 + 5$$

$$\chi^2 = 16.67$$

Number of degrees of freedom $= (2-1)(2-1) = 1$

Critical value:

The table value of χ^2 at 5% Los, for 1 d.f is 3.841

Conclusion:

Since $\chi^2 > 3.841$, H_0 is rejected at 5% Los.

$\therefore$ We conclude that the nature of work is dependent on sex of the worker.

4. From the following data, test whether there is any association between intelligency and

economics conditions

Intelligences						
		Excellent	Good	Medium	Dull	Total
Economic	Good	48	200	150	80	478
Conditions	Not Good	52	180	190	100	522
	Total	100	380	340	180	1000

Solution:

H_0 : There is no association between intelligency and economic conditions.

Los : $\alpha = 0.05$ (or) 5%

Under H_0, the expected frequencies are obtained as follows

$E(48) = \dfrac{100 \times 478}{1000} = 47.8;$ $E(52) = \dfrac{100 \times 522}{1000} = 52.2$

$E(200) = \dfrac{380 \times 478}{1000} = 181.64;$ $E(180) = \dfrac{380 \times 522}{1000} = 198.36$

$E(150) = \dfrac{478 \times 340}{1000} = 162.52;$ $E(190) = \dfrac{340 \times 522}{1000} = 177.48$

$E(80) = \dfrac{180 \times 478}{1000} = 86.04;$ $E(100) = \dfrac{180 \times 522}{1000} = 93.96$

O_i	:	48	200	150	80	52	180	190	100
E_i	:	47.8	181.64	162.52	86.04	52.2	198.36	177.48	93.96

$\therefore$ Test Statistic χ^2

$$\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i^{\ 2}}{E_i}$$

$$= \frac{48 - 47.8^{\ 2}}{47.8} + \frac{150 - 162.52^{\ 2}}{162.52} + \frac{52 - 52.2^{\ 2}}{52.2} + \frac{190 - 177.48^{\ 2}}{177.48} + \frac{200 - 181.64^{\ 2}}{181.64}$$

$$+ \frac{80 - 86.04^{\ 2}}{86.04} + \frac{180 - 198.36^{\ 2}}{198.36} + \frac{100 - 93.96^{\ 2}}{93.96}$$

$= 0.0008 + 0.9645 + 0.0008 + 0.8832 + 1.8558 + 0.4240 + 1.6994 + 0.3883$

$\quad \chi^2 = 6.2168$

Number of degrees of freedom $= (s-1)(t-1) = (2-1)(4-1) = 3$

Critical value:

The table value of χ^2 at 5% Los for 3 d.f is 7.815

Conclusion:

Since $\chi^2 < 7.815$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that there is no association between intelligency and economic conditions

5. From the following data, test the hypothesis that the flower color is independent of flatness

of leaf

	Flat leaves	Curved leaves	Total
White Flowers	99	36	135
Red Flowers	20	5	25
Total	119	41	160

Solution:

We set up: H_0 : flower color is independent of flatness of leaf.   Los α=0.05 (or) 5%

The given probelm is a 2 x 2 consistency table

$\therefore$ we use the formula to find χ^2 is

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(a+c)(c+d)(b+d)}$$

Here, a = 99;   b = 36;   e = 20;   d = 5

$$\chi^2 = \frac{160(495-720)^2}{(135)(119)(25)(41)} = \frac{160(50625)}{16,466,625}$$

$$\chi^2 = 0.4919$$

Number of degrees of freedom   = (s-1)(t-1) = (2-1)(2-1) = 1

Critical value:

The table value of χ^2 at 5% Los for 1 d.f is 3.841

Conclusion:

Since $\chi^2 < 3.841$, H_0 is accepted at 5% Los.

$\therefore$ Flower colour is independent of flatness of leaf.

Test for single variance

Chi-square test for population variance

In this method, we set up the null hypothesis $H_0 : \sigma^2 = \sigma_0{}^2$ (with a specified variance)

The test statistic $\quad \chi^2 = \dfrac{ns^2}{\sigma^2}$

Where n = sample size

        s = sample variance

        σ = population variance

Note:

    $*$ If the sample size n is large (>30)

    The test statistic $\quad z = \sqrt{2\chi^2} - \sqrt{2n-1} \sim N(0,1)$

We use the usual normal test.

1. A random sample of size 9 from a normal population have the following values 72, 68, 74, 77, 61, 63, 63, 73, 71. Test the hypothesis that the population variance is 36.

Solution:

    Null hypothesis $H_0 : \sigma^2 = 36$

    Alternative hypothesis $H_1 : \sigma^2 \neq 36$

    Los α : 0.05 (or) 5%

    $\therefore$ The test statistic $\quad \chi^2 = \dfrac{ns^2}{\sigma^2}$

    x : 72   68   74   77   61   63   63   73   71

$$\sum x = 622; \quad \bar{x} = \frac{\sum x}{n} = \frac{622}{9} = 69.11$$

| $x - \bar{x}$: | 2.9 | -1.1 | 4.9 | 7.9 | -8.1 | -6.1 | -6.1 | 3.9 | 1.9 |

| $\overline{x - \bar{x}}^2$: | 8.41 | 1.21 | 24.01 | 62.41 | 65.61 | 37.21 | 37.21 | 15.21 | 3.61 |

$$\sum \overline{x - \bar{x}}^2 = 254.89$$

$$\chi^2 = \frac{ns^2}{\sigma^2} = \frac{254.89}{36} = 7.08$$

d. f   n-1 = 9-1 = 8

Critical value:

The table value of χ^2 for 8 d.f at 5% Los is 15.51

Conclusion:

Since $\chi^2 < 15.51$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that the hypothesis of population variance is 36 is accepted

2. Test the hypothesis that $\sigma = 10$, given that s = 15 for a random sample of size 50 from a

normal population

Solution:

Null hypothesis $H_0 : \sigma = 10$

Alternative hypothesis $H_1 : \sigma \neq 36$

We are given n = 50;   s = 15

$$\chi^2 = \frac{ns^2}{\sigma^2} = \frac{50 \times 225}{100} = 112.5$$

Since 'n' is large (n > 30, the test statistic $z = \sqrt{2\chi^2} - \sqrt{2n-1}$

$$= \sqrt{225} - \sqrt{99} = 15 - 9.95$$

$$z = 5.05$$

This statistic z follows N(0,1)

Critical value:

At 5% Los, the table value of z is 3

Conclusion:

Since $|z| > 3$, H_0 is rejected.

$\therefore$ We conclude that $\sigma \neq 10$

3. The standard deviation of the distribution of times taken by 12 workers for performing a Job

is 11 sec. Can it be taken 1 as a sample from a popualation whose S.D is 10 sec.

Solution:

Let $H_0 : \sigma = 10$

ie., the population standard deviation $\sigma = 10$

$H_1 : \sigma \neq 10$

Los α : 0.05 (or) 5% Los

Given n = 12;   s = 11

$\therefore$ The test statistic is

$$\chi^2 = \frac{ns^2}{\sigma^2}$$

$$= \frac{12 \times 121}{100} = 14.52$$

$$\chi^2 = 14.52$$

Degrees of freedom = n - 1 = 12 - 1 = 11

Critical value:

The table value of χ^2 for 11 d.f at 5%Los is 19.675.

Conclusion:

Since $\chi^2 < 19.675$, H_0 is accepted at 5% level

$\therefore$ The S.D of the time element is 10 sec is supported.

ie., the population standard deviation $\sigma = 10$

UNIT-II

DESIGN OF EXPERIMENTS

Analysis of variance:

The technique of analysis of variance is referred to as ANOVA. A table showing the source of variance, the sum of squares, degrees of freedom, mean squares(variance)and the formula for the "F ratio is known as ANOVA table"

The technique of analysis if variance can be classified as

(i) One way classification(CRD)
(ii) Two way classification(RBD)
(iii) Three way classification(LSD)

One way classification:

In one way classification the data are classified on the basic of one criterion

The following steps are involved in one criterion of classification

(i) The null hypothesis is

$$H_o : \mu_1 = \mu_2 = ... = \mu_k$$

$$H_1 : \mu_1 \neq \mu_2 \neq .... \neq \mu_k$$

(ii) Calculation of total variation

Total sum of squares $V = \sum_i \sum_j x_{ij}^2 - \dfrac{G^2}{N}$

Where $G = \sum_i \sum_j x_{ij}$ (Grand total)

$\dfrac{G^2}{N}$ =correction formula

(iii) Sum of squares between the variates

$$V_1 = \sum_i \left[\dfrac{T_i^2}{n_i} \right] - \dfrac{G^2}{N} \text{ With k-1 degree of freedom}$$

(iv) Sum of squares within samples

$$V_2 = V - V_1$$

then the ratio $\dfrac{\dfrac{V_1}{K-1}}{\dfrac{V_2}{N-K}}$ follows F-distribution with degrees of freedom. Choosing the ratio which is

greater than one, we employ the F-test

If we calculated F < table value $F_{0.05}$, the null hypothesis is accepted.

ANOVA Table for one way classification

Source of variation	Sum of square	Degrees of freedom	Mean square	Variance ratio
Between classes	V_1	K-1	$\dfrac{V_1}{K-1}$	$\dfrac{\dfrac{V_1}{K-1}}{\dfrac{V_2}{N-K}}$ (or)
Within classes	V_2	$N-K$	$\dfrac{V_2}{N-K}$	$\dfrac{\dfrac{V_2}{N-K}}{\dfrac{V_1}{K-1}}$
	V	$N-1$		

1. To test the significance of the variation of the retail prices of a certain commodity in the four principal plates A,B,C &D, seven shops were chosen at random in each city and the prices observed were as follows (prices in paise)

A	82	79	73	69	69	63	61
B	84	82	80	79	76	68	62
C	88	84	80	68	68	66	66
D	79	77	76	74	72	68	64

Do the data indicate that the prices in the four cities are significantly different?

Solution:

$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$

$H_1 : \mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$

i.e., the prices of commodity in the four cities are same.

we take the origin at $x = 80$ and the calculation are done as follows.

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Calculation of ANOVA (use new values)

Cities K=4	Shop(n = 7) 1	2	3	4	5	6	7	T_i	$\dfrac{T_i^2}{n}$	$\sum x^2$
A	2	-1	-7	-11	-11	-17	-19	-64	585.14	946
B	4	2	0	-1	-4	-12	-18	-29	120.14	505
C	8	4	0	-12	-12	-14	-14	-40	228.57	760
D	-1	-3	-4	-6	-8	-12	-16	-50	357.14	526
$\dfrac{G^2}{N}=1196.03$								$G=-183$	$\dfrac{\sum T_i^2}{n}=1290.9$	$\sum_i \sum_j x_{ij}^2 = 2737$

Total sum of squares $V = \sum_i \sum_j x_{ij}^2 - \dfrac{G^2}{N}$

$$=2737-1196.03$$

$$V=1540.97$$

Sum of squares between cities

$$V_1 = \sum \frac{T_i^2}{n} - \frac{G^2}{N}$$

$$= 1290.9 - 1196.03$$

$$V_1 = 94.87$$

Sum of squares within cities

$$V_2 = V\text{-}V_1 = 1540.97 - 94.87$$

$$V_2 = 1446.1$$

ANOVA Table:

Source of variation	Sum of square of deviation	Degrees of f	Mean square	F
Between cities	$V_1 = 94.87$	K-1=4-1=3	$\dfrac{V_1}{K-1}=\dfrac{94.87}{3}$ $=31.62$	$=\dfrac{60.25}{31.62}$ $=1.90$
Within cities	$V_2 = 1446.1$	N-K=28-4=24	$\dfrac{V_2}{N-K}=\dfrac{1446.1}{24}$ $=60.25$	
Total	V=1540.97	N-1=27		

Number of degrees of freedom = (N - K, K - 1) = (24,3)

Critical value:

The table value of F for (24, 3) degree of freedom at 5% Los is 8.64

Conclusion:

Since F< 8.64, H_o is accepted at 5% Los

∴ The prices of commodity in the four cities are same

2. Fill up the following Analysis of variance table

Source of variation	Degrees of freedom	Sum of squares	Mean squares	F ratio
Treatments	-	-	117	
Error	-	704	-	-
Total	16	938		

Solution:

From the given table we have,

$$V_2 = 704; V = 938$$

degree of freedom (total) N - 1 = 16 => N = 17

mean squares $\dfrac{V_1}{K-1} = 117$

We Know that $V_2 = V - V_1$

$$\Rightarrow V_1 = V - V_2$$

$$= 938 - 704$$

$$\boxed{V_1 = 234}$$

$$\frac{V_1}{K\text{-}1} = 117$$

$$\Rightarrow \frac{234}{K\text{-}1} = 117 \Rightarrow \frac{234}{K\text{-}1} = K\text{-}1$$

$$K - 1 = 2$$

degree of freedom (K-1) = 2

=>K=3

Next, N-K = 17-3 = 4

$$\frac{V_2}{N\text{-}K} = \frac{938}{14} = 50.29$$

Source of variation	Degrees of freedom	Sum of squares	Mean squares	F ratio
Treatments	K-1=3-1=2	$V_1 = 234$	$\frac{V_1}{K\text{-}1} = 117$	$\frac{117}{50.29}$
Error	N-K=17-3=14	$V_2 = 704$	$\frac{V_2}{N\text{-}K} = 50.29$	$= 2.327$
Total	16	V = 938		

3.   The following are the number of mistakes made in 5 successive days of 4 technicians working in a photographic laboratory

Technicians I	Technicians II	Technicians III	Technicians IV
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11

Test at the 1% Los whether the difference among the 4 samples means can be attributed to chance

Solution:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$

ie., There is no differences among the 4 samples mean

$$H_1 : \mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$$

We take the origin at 12 and the calculation are done as follows

Calculation of ANOVA (NEW Values)

Technicians $K = 4$	Days(5)					T_i	$\dfrac{T_i^2}{n}$	$\sum x^2$
	1	2	3	4	5			
I	-6	2	-2	-4	-1	-11	24.2	61
II	2	-3	0	-2	2	-1	0.2	21
III	-2	0	-5	3	-1	-5	5	39
IV	-3	0	-4	-2	-1	-10	20	30
Total	$\dfrac{G^2}{N} = \dfrac{(-27)^2}{20} = 36.45$					G=-27	49.4	151

Total sum of squares:

$$V = \sum_i \sum_j x_{ij}^2 - \frac{G^2}{N}$$

$$= 151 - 36.45$$

$$V = 114.55$$

Sum of squares b/w cities:

$$V_1 = \sum \frac{T_i^2}{n} - \frac{G^2}{N}$$

$$= 49.4 - 36.45$$

$$V_1 = 12.95$$

Sum of squares within cities:

$$V_2 = V - V_1 = 114.55 - 12.95$$

$$V_2 = 101.6$$

Source of variation	Sum of squares of deviation	Degrees of freedom	Mean squares	F ratio
B/W Technicians	$V_1 = 12.95$	K-1=4-1=3	$\dfrac{V_1}{K\text{-}1} = \dfrac{12.95}{3} = 4.31$	$= \dfrac{6.35}{4.31}$
Within Technicians	$V_2 = 101.6$	N-K=20-4=16	$\dfrac{V_2}{N\text{-}K} = \dfrac{101.6}{16} = 6.35$	
Total	V=114.55	N-1=19		$= 1.473$

Degrees of freedom ((N - K, K - 1) = (16,3)

Critical value:

The table value of 'F' for (16,3) degree of freedom at 1% Los is 5.29

Conclusion:

Since F < 5.29, H_0 accepted at 1% level

$\therefore$ There is no difference among the four sample means.

4. The following table shows the lives in hours of four batches of electric lamps.

Batches	Lives in hours							
1	1610	1610	1650	1680	1700	1720	1800	
2	1580	1640	1640	1700	1750			
3	1460	1550	1600	1620	1640	1660	1740	1820
4	1510	1520	1530	1570	1600	1680		

Perform an analysis of the variance on these data and show that a significant test does not reject their homogeneity

Solution:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$

I.e., the means of the lives of the four brands are homogeneous.

$$H_1 : \mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$$

We take the origin $x_{ij} = \dfrac{\text{old}\,x_{ij} - 1700}{10}$

Calculation of ANOVA

Brand K=4	Lives								T_i	$\dfrac{T_i^{\,2}}{n}$	$\sum\limits_{ij} x_{ij}$
	1	2	3	4	5	6	7	8			
1	-9	-9	-5	-2	0	2	10	-	-13	24.143	295
2	-12	-6	-6	0	5	-	-	-	-19	72.2	241
3	-24	-15	-10	-8	-6	-4	4	12	-51	325.125	1177
4	-19	-18	-17	-13	-10	-2	-	-	-79	1040.167	1247
Total	$\dfrac{G^2}{N} = \dfrac{(-162)^2}{26} = 1009.38$								G=-162	=1461.635	2960

$N = n_1 + n_2 + n_3 + n_4 = 7 + 5 + 8 + 6 = 26$

Total sum of squares:

$$V = \sum_i \sum_j (x_{ij})^2 - \frac{G^2}{N}$$

$$= 2960 - 1009.38$$

$V = 1950.62$

Total sum of squares b/w brands:

$$V_1 = \sum \frac{T_i^{\,2}}{n} - \frac{G^2}{N}$$

$$= 1461.635 - 1009.38$$

$V_1 = 452.255$

Sum of squares within brands:

$$V_2 = V - V_1$$

$$= 1950.62 - 452.255$$

$V_2 = 1498.365$

ANOVA Table:

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
B/W Brands	$V_1 = 452.255$	K-1=4-1=3	$\dfrac{V_1}{K\text{-}1} = \dfrac{452.255}{3} = 150.75$	$= \dfrac{150.75}{68.11}$ $= 2.21$
Within Brands	$V_2 = 1498.365$	N-K=26-4=22	$\dfrac{V_2}{N\text{-}K} = \dfrac{1498.365}{22} = 68.11$	
Total	$1950.62 = V$	N-1=25		

Degrees of freedom (3, 22) =3.05

Critical value:

The table value of 'F' for (3,22) d.f at 5% Los is 3.05

Conclusion:

Since F<3.05, H_0 is accepted at 5% level

$\therefore$ The means of the lives of the four brands are homogeneous.

ie., the lives of the four brands of lamps do not differ significantly.

Two way classification:

In two way classification the data are classified on the basis of two criterions

The following steps are involved in two criterion of classification

(i) The null hypothesis

H_{01} and H_{02} framed

We compute the estimates of variance as follows

(ii) $G = \sum_i \sum_j x_{ij} =$ Grand total of $K \times n$ Observations

(iii) S : Total sum of squares $\sum \sum x_{ij}^2 - \dfrac{G^2}{N}$

(iv) S_1 :Sum of squares b/w rows (class-B) $= \dfrac{1}{K} \sum\limits_{j=1}^{n} R_j^2 - \dfrac{G^2}{N}$

(v) S_2 :Sum of squares b/w (classes A) $= \dfrac{1}{n} \sum\limits_{i=1}^{K} C_i^2 - \dfrac{G^2}{N}$

S_3 : Sum of squares due to error (or) Residual sum of squares

(vi)   Errors (or) Residual $S_3 = S - S_1 - S_2$

(vii) The degrees of freedoms of

$S_1 = n\text{-}1 \ ; S_2 = k\text{-}1 \ ; S_3 = (n\text{-}1)(k\text{-}1)$

$S = nk\text{-}1$

ANOVA Table for two way classification

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
B/W 'B' classes(rows)	S_1	n-1	$\dfrac{S_1}{n\text{-}1} = Q_B$	$F_1 = \dfrac{Q_B}{Q_{AB}}$ $d.f = [(n\text{-}1)(k\text{-}1)(n\text{-}1)]$
B/W 'A' classes(column)	S_2	k-1	$\dfrac{S_2}{k\text{-}1} = Q_A$	$F_2 = \dfrac{Q_A}{Q_{AB}}$
Residual (or) error	S_3	(n-1)(k-1)	$\dfrac{S_3}{(n\text{-}1)(k\text{-}1)} = Q_{AB}$	$d.f = [(k\text{-}1),(k\text{-}1)(n\text{-}1)]$
Total	S	nk-1	-	-

Advantages of R.B.D:

The chief advantages of R.B.D are as follows

(i)   This design is more efficient or more accurate than CRD. This is because of reduction of experimental error.

(ii)   The analysis of the design is simple and even with missing observations, it is not much complicated

(iii) It is Quite flexible, any number of treatments and any number of replication may be used

(iv)   It is easily adaptable as in agricultural experiment it can be accommodated well in a rectangular, squares(or)in a field of any shape

(v)   It provides a method of eliminating or reducing the long term effects.

(vi)   This is the most popular design with experiments in view of its simplicity, flexibility and validity. No other has been used so frequently as the R.B.D

Disadvantages:

- (i) The number of treatments is very large, than the side of the blocks will increase and this may introduce heterogeneity within blocks.
- (ii) If the interactions are large, the experiments may yield misleading results.

1. The following data represent the number of units of production per day turned out by four randomly chosen operators using three milling machines

Machines.

		M_1	M_2	M_3
Operators	1	150	151	156
	2	147	159	155
	3	141	146	153
	4	154	152	159

Perform analysis of variance and test the hypothesis

- (i) That the machines are not significantly different
- (ii) That the operators are not significantly different at 5% level

Solution:

H_{01} : There is no significantly difference bet machine and

H_{02} : There is no significantly a difference b/w operator

We take the origin 155 and the calculations are done as follows.

Calculation of ANOVA (using new values)

Operators	Machines			Row total R_j	$\sum_j x_{ij}^2$
	M1	M2	M3		
1	-5	-4	1	-8	42
2	-8	4	0	-4	80
3	-14	-9	-2	-25	281
4	-1	-3	4	0	26
Column total C_i	-28	-12	3	-37	429
$\sum_i x_{ij}^2$	286	122	21	429	

Here N=12 ; G=-37

Correction factor $\dfrac{G^2}{N} = \dfrac{(-37)^2}{12} = 114.08$

Total sum of squares:

$$S = \sum_i \sum_j x_{ij}^2 - \frac{G^2}{N}$$

$$= 429 - 114.08$$

$$= 314.92$$

Sum of squares between operators:

$$S_1 = \sum_j \frac{R_j^2}{n_j} - \frac{G^2}{N}$$

$$= \frac{1}{3}[(-8)^2 + (-4)^2 + (-25)^2] - 114.08$$

$$= 235 - 114.08$$

$$= 120.92$$

Sum of squares between machines:

$$S_2 = \sum_i \left(\frac{C_i^2}{n_i} \right) - \frac{G^2}{N}$$

$$= \frac{1}{4}\left[(-28)^2 + (-12)^2 + (3)^2 \right] - 114.08$$

$$= 234.25 - 114.08$$

$$S_2 = 120.17$$

Residual sum of squares:

$$S_3 = S - S_1 - S_2$$

$$= 314.92 - 120.92 - 120.17$$

$$= 73.83$$

AVOVA Table for two way classification

Source of variation	Sum of squares	Degrees of freedom	Mean sum squares	F ratio
B/W operators	120.92	n-1=4-1=3	$Q_B = \dfrac{S_1}{n-1} = 40.31$	
B/W machines	120.17	k-1=3-1=2	$Q_A = \dfrac{S_2}{k-1}$ $= 60.09$	$\dfrac{40.31}{12.305} = 1.49$ (3,6)
Residual	73.83	(n-1)(k-1)=6	$Q_{AB} = \dfrac{S_3}{(k-1)(n-1)}$ $= 12.305$	$\dfrac{60.09}{12.305} = 4.88$ (2,6)
Total	314.92	nk-1=11		

Degrees of freedom $V_1 = 2; V_2 = 6$ (machines)

Degrees of freedom $V_1 = 3; V_2 = 6$ (operators)

Critical value:

(i) Machines

The table value of 'F' for (2,6) d.f at 5% Los is 5.14

(ii) Operators

The table value of 'F' for (3,6) d,f at 5% Los is 4.76

Conclusion:

(i) Operators

Since F< 4.76 , H_{02} is accepted at 5% level

∴ The operators are not significantly different

(ii) For Machines

Since F< 5.14, H_{01} is accepted at 5% level

∴ The machines are not significantly different

2. An experiment was designed to study then performance of four different detergents, the following "whiteness" readings were obtained with specially designed equipment for 12 loads of washing distributed over three different models of washing machines.

Machines / Detergents	1	2	3	Total
A	45	43	51	139
B	47	46	52	145
C	48	50	55	153
D	42	37	49	128
Total	182	176	207	565

Looking on the detergents as treatment and the machines as blocks, obtain the appropriate analysis of variance table and test at 0.01 level of Significance whether there are differences in the detergents (or) in the washing machines

Solution:

H_{01} : There is no significant different b/w detergent

H_{02} : There is no significant different b/w washing machine

We take the origin is 50 and the calculation are done as follows.

Calculation of ANOVA (using new values)

Detergents	Washing machines			Row total R_j	$\sum_j x_{ij}^2$
	M1	M2	M3		
A	-5	-7	1	-11	75
B	-3	-4	2	-5	29
C	-2	0	5	3	29
D	-8	-13	-1	-22	234
Column total C_i	-18	-24	7	-35	367
$\sum_i x_{ij}^2$	102	234	31	367	

Here N=12; G=-35

Correction factor $\dfrac{G^2}{N} = \dfrac{(-35)^2}{12} = 102.08$

Total sum of squares: $S = \sum_i \sum_j x_{ij}^2 - \dfrac{G^2}{N}$

$$= 367 - 102.08$$

$$S = 264.92$$

Sum of squares b/w detergents: $S_1 = \sum_j \dfrac{R_j^2}{h_j} - \dfrac{G^2}{N}$

$$= \frac{1}{3}\left[(-11)^2 + (-5)^2 + (3)^2 + (-22)^2\right] - 102.08$$

$$= 213 - 102.08$$

$$S_1 = 110.92$$

Sum of squares between machines

$$S_2 = \sum_i \left(\frac{C_i^2}{n_i}\right) - \frac{G^2}{N}$$

$$= \frac{1}{4}\left((-18)^2 + (-24)^2 + (7)^2\right) - 102.08$$

$$= 237.25 - 102.08$$

$$S_2 = 135.17$$

Residual sum of squares $S_3 = S - S_1 - S_2$

$$= 264.92 - 110.92 - 135.17$$

$$S_3 = 18.83$$

ANOVA table for two way classification:

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
B/W detergents	$S_1 = 110.92$	n-1=4-1=3	$Q_B = \dfrac{S_1}{n\text{-}1} = \dfrac{110.92}{3}$ $= 36.97$	$\dfrac{Q_B}{Q_{AB}} = \dfrac{36.97}{3.14}$ $= 11.77$
B/W machines	$S_2 = 135.17$	k-1=3-1=2	$Q_A = \dfrac{S_2}{k\text{-}1} = \dfrac{135.17}{2}$ $= 67.59$	$\dfrac{Q_A}{Q_{AB}} = \dfrac{67.59}{3.14}$ $= 21.52$
Residual (or) Error	$S_3 = 18.83$	(n-1)(k-1)=6	$Q_{AB} = \dfrac{S_3}{(n\text{-}1)(k\text{-}1)} = \dfrac{18.83}{6}$ $= 3.14$	
Total	S=264.92	nk-1=11		

Degrees of freedom $V_1 = 2; V_2 = 6$ (machines)

Degrees of freedom $V_1 = 3; V_2 = 6$ (detergents)

Critical value:

(i) Detergents:

The table value of F for (3,6) degree of freedom at 1% Los is 9.78

(ii) Machines

The table value of F for (2,6) degree of freedom at 1% Los is 10.92

Conclusion:

(i) For detergents

Since F>9.78, H_{01} is rejected at 5% level

∴ The detergents are significantly different

(ii) For machines

Since F>10.92, H_{02} is rejected at 5% level

∴ The machines are significantly different

3. To study the performance of three detergents and three different water temperatures the following whiteness readings were obtained with specially designed equipment.

Water temp	Detergents A	Detergents B	Detergents C
Cold Water	57	55	67
Worm Water	49	52	68
Hot Water	54	46	58

Solution:

We set the null hypothesis

H_{01} : There is no significant different in the three varieties of detergents

H_{02} : There is no significant different in the water temperatures

We choose the origin at x=50

Water temp	Detergents			Row total R_j	$\sum_j x_{ij}^2$
	A	B	C		
Cold Water	7	5	17	29	363
Worm Water	-1	2	18	19	329
Hot Water	4	-4	8	8	96
Column total C_i	10	3	43	56	788
$\sum_i x_{ij}^2$	66	45	677	788	

Total sum of squares:

$$S = \sum_j \sum_i x_{ij}^2 - \frac{G^2}{N}$$

$$= 788 - \frac{(56)^2}{9} = 788 - 348.44$$

S=439.56

Sum of squares between detergents:

$$S_1 = \sum_i \frac{C_i^2}{n_i} - \frac{G^2}{N}$$

$$= \frac{1}{3}\left[(10)^2 + (3)^2 + (43)^2\right] - 348.44$$

$$= 652.67 - 348.44$$

$$S_1 = 304.23$$

Sum of squares b/w temperatures:

$$S_2 = \sum_j \frac{R_j^2}{n_j} - \frac{G^2}{N}$$

$$= \frac{1}{3}[1266] - 348.44$$

$$= 422 - 348.44$$

$$S_2 = 73.56$$

Error sum of squares:

$$S_3 = S - S_1 - S_2$$

$$= 439.56 - 304.23 - 73.56$$

$$S_3 = 61.77$$

ANOVA Table:

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
B/W detergents	304.23	2	$\frac{304.23}{2} = 152.11$	$\frac{152.11}{15.445} = 9.848$ $(2,4)$
B/W temperatures	73.55	2	$\frac{73.56}{2} = 36.78$	$\frac{36.78}{15.445} = 2.381$
Error	61.79	4	15.445	
Total	439.56	8		

Degrees of freedom (2,4) and (2,4)

Critical value:

The table value of F for (2,4) d.f at 5% Los is 6.94

Conclusion:

(i)   For detergents:

Since $F > 9.85$, H_{01} is rejected at 5% Los

∴ There is a significant different between the three varieties detergents,

(iii)   For water temperature

Since $F < 6.94$, H_{02} is accepted at 5% Level

∴ There is no significant different in the water temperatures.

4.   Four experiments determine the moisture content of samples of a powder, each man taking a sample from each of six consignments. These assignments are

Observer	Consignment					
	1	2	3	4	5	6
1	9	10	9	10	11	11
2	12	11	9	11	10	10
3	11	10	10	12	11	10
4	12	13	11	14	12	10

Perform an analysis if variance on these data and discuss whether there is any significant different b/w consignments (or) b/w observers.

Solution:

We formulate the hypothesis

H_{02} : There is no significant different b/w observer

H_{02} : There is no significant different b/w consignment

We take origin at x=11 and the calculations are done are as follows

Calculation ANOVA:

Observer	consignments						Rowtotal R_j	$\sum_j x_{ij}^2$
	1	2	3	4	5	6		
1	-2	-1	-2	-1	0	0	-6	10
2	1	0	-2	0	-1	-1	-3	7
3	0	-1	-1	1	0	-1	-2	4
4	1	2	0	3	1	-1	6	16
Column total C_i	0	0	-5	3	0	-3	-5	37
$\sum_j x_{ij}^2$	6	6	9	11	2	3	37	

Total sum of squares $= \sum_j \sum_i x_{ij}^2 - \frac{G^2}{N}$

$$S = 37 - \frac{(-5)^2}{24} = 35.96$$

Sum of squares b/w observers $= \sum \frac{(R_j)^2}{n_j} - \frac{G^2}{N}$

$$S_1 = \frac{1}{6}\left[(-6)^2 + (-3)^2 + (-2)^2 + (6)^2\right] - \frac{25}{24}$$

$$S_1 = 13.13$$

Sum of squares b/w consignments $= \sum \left(\frac{C_i^2}{n_i}\right) - \frac{G^2}{N}$

$$S_2 = \frac{1}{4}\left[(0 + 0 + 25 + 9 + 9)\right] - \frac{25}{24}$$

$$S_2 = 9.71$$

Error sum of squares $S_3 = S - S_1 - S_2$

$$= 35.96 - 13.13 - 9.71$$

$$S_3 = 13.12$$

Source of variation	Sum of squares	Degrees of freedom	Mean squares	'F' ratio
B/W Consignments	$S_1 = 9.71$	n-1=5	$\frac{9.71}{5}$ $= 1.94$	$\frac{1.94}{0.87}$ $= 2.23$ (5,15)
B/W observers	$S_2 = 13.13$	k-1=3	$\frac{13.13}{3}$ $= 4.38$	$\frac{4.38}{0.87}$ $= 5.03$
Error	$S_3 = 13.12$	(n-1)(k-1)=15	$\frac{13.12}{15}$ $= 0.87$	(3,15)
Total	$S = 35.96$	nk-1=23		

Critical value:

(i) For consignments ,

The table value of 'F' for (5, 15) d.f at 5% Los is 2.90

(ii) For observers:

The table value of F for (3, 15) d,f at Los 3.29

Conclusion:

(i) For observers

Since F>3.29, H_{01} is rejected

Hence these is a difference between observers is significant

(ii) For consignment:

Since F<2.33, H_{02} is accepted

$\therefore \therefore$ There is no significant different b/w the consignments

LATIN SQUARES DESIGN:

A Latin squares is a squares arrangement of m-rows and m-columns such that each symbol appearly once and only once in each row and column.

In randomized block design the randomization is done within blocks the units in each block being relatively similar in L.S.D there are two restrictions

(i) The number of rows and columns are equal

(ii) Each treatment occurs once and only once in each row and column.

This design is a three way classification model analysis of variance

The following steps are involved in Latin square design

Correction factor $= \dfrac{G^2}{N};$ G -> Grand total

$$\text{S.S b/w rows=}S_a = \sum_{i=1}^{m} \frac{S_i^2}{m} - C.F \qquad \text{(S.S means Sum of Squares)}$$

$$\text{S.S b/w Columns=}S_b = \sum_{j=1}^{m} \frac{S_j^2}{m} - \frac{G^2}{N} |C.F$$

$$\text{S.S b/w Varieties=}S_c = \sum_{i=1}^{m} \frac{V_i^2}{m} - C.F$$

$$\left.\begin{array}{c}\textit{Total sum of}\\\textit{squares}\end{array}\right\} S = \sum_j \sum_i x_{ij}^2 - C.F$$

$$\text{and } S_d = S - S_a - S_b - S_c$$

Here S_i = sum of ith row

S_j = sum of jth column

V_i = sum of ith variety

ANOVA Table:

Source of variation	Sum of squares	Degrees of freedom	Mean squares	'F' ratio
B/W Rows	S_a	m-1	$\dfrac{S_a}{m-1} = R$	$\dfrac{R}{E}$ $\left[(m\text{-}1),(m\text{-}1)(m\text{-}2)\right]$
B/W Columns	S_b	m-1	$\dfrac{S_b}{m-1} = C$	$\dfrac{C}{E}$ $\left[(m\text{-}1),(m\text{-}1)(m\text{-}2)\right]$
B/W varieties	S_c	m-1	$\dfrac{S_c}{m-1} = V$	$\dfrac{V}{E}$ $\left[(m\text{-}1),(m\text{-}1)(m\text{-}2)\right]$
Error	S_d	(m-1)(m-2)	$\dfrac{S_d}{(m\text{-}1)(m\text{-}2)} = E$	
Total	S	$m^2 - 1$		

Comparison of LSD and RBD

(i) In LSD, the number of rows and number of columns are equal and hence the number of replication is equal to the number of treatments there is no such restriction in RBD

(ii) L.S.D is suitable for the case when the number of treatments is b/w 5 and 12 where as R.B.D can be used for any number of treatments and replications

(iii) The main advantage of L.S.D is that it removes the variations b/w rows and columns from that within the rows resulting in the reduction of experiment error to a large extent

(iv) The RBD can be performed equally on rectangular of square plots but for LSD, a mose (or) less a squares field is required due to (iii) LSD is preferred over RBD

Note: A 2×2 Latin Square Design is not possible. The degree of freedom for error in a $m \times m$ Latin squares design is (m-1)(m-2)

For m=2 the degree of freedom is 'o' and hence comparisons are not possible.

Hence a 2×2 LSD is not possible.

1. The following is the LSD layout of a design when 4 varieties of seeds are being tested set up the analysis of variance table and state four conclusion

A 105	B 95	C 125	D 115
C 115	D 125	A 105	B 105
D 115	C 95	B 105	A 115
B 95	A 135	D 95	C 115

Solution:

H: There is no significant difference

we take the origin as $u_{ij} = \dfrac{x_{ij} - 100}{5}$ and the calculations are done as follows

Varieties	Values				V_i
A	1	1	3	7	12
B	-1	1	-1	-1	0
C	5	3	-1	3	10
D	3	5	3	-1	10

Columns / Rows	C_1	C_2	C_3	C_4	Row total R_j	$\sum_i x_{ij}^2$
R_1	1	-1	5	3	8	36
R_2	3	5	1	1	10	36
R_3	3	-1	1	3	6	20
R_4	-1	7	-1	3	8	60
Columns total C_i	6	10	6	10	G=32	152
$\sum_j x_{ij}^2$	20	76	28	28	152	

$G=32 \qquad N=16; \qquad \sum_j \sum_i x_{ij}^2 = 152$

$\text{C.F} = \dfrac{G^2}{N} = \dfrac{(+32)^2}{16} = 64$

Total sum of squares $= \sum_j \sum_i x_{ij}^2 - \dfrac{G^2}{N}$

$$= 152 - \dfrac{(32)^2}{16}$$

$$= 152 - 64$$

$$S = 88$$

Sum of squares b/w rows $= \dfrac{1}{4}\left[8^2 + 10^2 + 6^2 + 8^2\right] - 64$

$$= 66 - 64$$

$$S_a = 2$$

Sum of squares b/w columns $= \dfrac{1}{4}\left[6^2 + 10^2 + 6^2 + 10^2\right] - 64$

$$S_b = 68 - 64$$

$$S_b = 4$$

Sum of squares b/w Varieties $= \dfrac{1}{4}\left[12^2 + 0^2 + 10^2 + 10^2\right] - 64$

$$= 86 - 64$$

$$S_c = 22$$

Error sum of squares $S_d = S - S_a - S_b - S_c$

$$= 88 - 2 - 4 - 22$$

$$S_d = 60$$

ANOVA Table:

Source of variation	Sum of squares	Degrees of freedom	Mean sum of squares	'F' ratio
B/W rows	$S_a = 2$	m-1=4-1=3	$\dfrac{S_a}{m\text{-}1} = \dfrac{2}{3} = 0.67$	$\dfrac{0.67}{10} = 0.067$
B/W columns	$S_b = 4$	m-1=4-1=3	$\dfrac{S_b}{m\text{-}1} = \dfrac{4}{3} = 1.33$	$\dfrac{1.33}{10} = 0.133$
B/W varieties	$S_c = 22$	m-1=3	$\dfrac{S_c}{m\text{-}1} = \dfrac{22}{3} = 7.33$	$\dfrac{7.33}{10} = 0.733$
Error	$S_d = 60$	(m-1)(m-2) $=3\times 2=6$	$\dfrac{S_d}{(m\text{-}1)(m\text{-}2)} = 10$	-
Total	$S = 88$	$m^2 - 1 = 15$	-	-

Number of degrees of freedom $V_1 = 3$; $V_2 = 6$

Critical value:

The table value of F for (3, 6) d.f at 5% Los is 4.76

Conclusion:

Since F<4.76, for all the case.

$\therefore$ There is no significant difference for the varieties

2. Analyse the variance in the following Latin squares of fields (in keys) of paddy where A,B,C,D denote the difference methods of calculation

D122	A121	C123	B122
B124	C123	A122	D125
A120	B119	D120	C121
C122	D123	B121	A122

Examine whether the different methods of cultivation have given significantly different fields.

Solution:

Re arrange the table in order

A121	A122	A120	A122
B122	B124	B119	B121
C123	C123	C121	C122
D122	D125	D120	D123

We take the origin 122 and the table is

Letter	Values				V_i total
A	-1	0	-2	0	-3
B	0	2	-3	-1	-2
C	1	1	-1	0	1
D	0	3	-2	1	2

Calculation of LSD:

Columns / Rows	1	2	3	4	Row total	$\sum_j x_{ij}^2$
1	0	-1	1	0	0	2
2	2	1	0	3	6	14
3	-2	-3	-2	-1	-8	18
4	0	1	-1	0	0	2
Columns total	0	-2	-2	2	-2	36
$\sum_i x_{ij}^2$	8	12	6	10	36	

Here N=16; G=-2

Correction factor = $\dfrac{G^2}{N} = \dfrac{4}{16} = 0.25$

Total sum of squares $S = \sum_i \sum_j x_{ij}^2 - \dfrac{G^2}{N}$

$$= 36 - 0.25$$

$$S = 35.75$$

Sum of squares b/w rows $S_a = \sum_{i=1}^{m} \dfrac{S_i^2}{m} - \dfrac{G^2}{N}$

$$= \frac{1}{4}\left[(6)^2 + (-8)^2\right] - 0.25$$

$$= 25 - 0.25$$

$$S_a = 24.75$$

Sum of squares b/w columns $S_b = \sum_{j=1}^{m} \frac{S_j^2}{m} - \frac{G^2}{N}$

$$= \frac{1}{4}\left[(0)^2 + (-2)^2 + (-2)^2 + (2)^2\right] - 0.25$$

$$S_b = 2.75$$

Sum of squares b/w varieties $S_c = \sum_{i=1}^{m} \frac{V_i^2}{m} - \frac{G^2}{N}$

$$= \frac{1}{4}\left[(-3)^2 + (-2)^2 + (1)^2 + (2)^2\right] - 0.25$$

$$= 4.5 - 0.25$$

$$S_c = 4.25$$

Error (or) Residual $S_d = S\text{-}S_a - S_b - S_c$

$$= 35.75 - 24.75 - 2.75 - 4.25$$

$$S_d = 4$$

LSD Table:

Source of variation	Sum of squares	Degrees of freedom	Mean sum of squares	'F' ratio
B/W rows	$S_a = 24.75$	m-1=3	$\frac{S_a}{m\text{-}1} = \frac{24.75}{3}$ $= 8.25$	$\frac{8.25}{0.67} = 12.31$
B/W columns	$S_b = 2.75$	3	$\frac{S_b}{m\text{-}1} = \frac{2.75}{3}$ $= 0.92$	$\frac{0.92}{0.67} = 1.37$
B/W varieties	$S_c = 4.25$	3	$\frac{S_c}{m\text{-}1} = \frac{4.25}{3}$ $= 1.42$	$\frac{1.42}{0.67} = 2.12$
Error (or) Residual	$S_d = 4.0$	6=(m-1)(m-2)	$\frac{S_d}{(m\text{-}1)(m\text{-}2)}$ $= 0.67$	
Total	$S = 35.75$	$m^2 - 1 = 8$		

com

Critical value:

The value of 'F' for (3,6) d.f at 5% Los is 4.76

Conclusion:

Since F<4.76, we accept the null hypothesis

∴ The difference between the methods of cultivation is not significant.

3. The following data resulted from an experiment to compare three burners A,B, and C,A Latin squares design was used as the tests were made on 3 engines and were spread over 3 days.

	Engine 1	Engine 2	Engine 3
Day 1	A 16	B 17	C 20
Day 2	B16	C 21	A 15
Day 3	C15	A 12	B 13

Test the hypothesis that there is no diff between the burners

Solution:

We take the origin x=15 and the calculation are done as follows

Re arrangement of given table is

A 16	B 17	C 20
A 15	B 16	C 21
A 12	B 13	C 15

Varieties	Values			V_i
A	1	0	-3	-2
B	2	1	-2	1
C	5	6	0	11

Calculation of LSD

Columns/ Rows	C_1	C_2	C_3	Row total	$\sum_j x_{ij}^2$
R_1	1	2	5	8	30
R_2	1	6	0	7	37
R_3	0	-3	-2	-5	13
Column total	2	5	3	10	80
$\sum_i x_{ij}^2$	2	49	29	80	

Here N=9; G=10

Correction Factor $= \dfrac{G^2}{N} = \dfrac{(10)^2}{9} = 11.11$

Total sum of squares $S = \displaystyle\sum_j \sum_i x_{ij}^{\ 2} - C.F$

$= 80 - 11.11$

$S = 68.89$

Sum of squares b/w Rows $S_a = \displaystyle\sum_{i=1}^{m} \dfrac{S_i^{\ 2}}{m} - C.F$

$= \dfrac{1}{3}[8^2 + 7^2 + (-5)^2] - 11.11$

$= 46 - 11.11$

$S_a = 34.89$

Sum of squares b/w columns $S_b = \displaystyle\sum_{j=1}^{m} \dfrac{S_j^{\ 2}}{m} - C.F$

$= \dfrac{1}{3}[(2)^2 + (5)^2 + (3)^2] - 11.11$

$= 1.56$

Sum of squares b/w varieties $S_c = \displaystyle\sum_{i=1}^{m} \dfrac{V_i^{\ 2}}{m} - C.F$

$= \dfrac{1}{3}[(-2)^2 + 1^2 + 11^2] - 11.11$

$S_c = 30.89$

Error (or) Residual $S_d = S - S_a - S_b - S_c$

$= 68.89 - 34.89 - 1.56 - 30.89$

$S_d = 1.55$

Source of variation	Sum of squares	Degrees of freedom	Mean sum of squares	'F' ratio
B/W rows	$S_a = 34.89$	m-1=2	$\dfrac{S_a}{m-1} = \dfrac{34.89}{2}$ $= 17.445$	$\dfrac{17.445}{0.775} = 22.5$
B/W columns	$S_b = 1.56$	m-1=2	$\dfrac{S_b}{m-1} = \dfrac{1.56}{2}$ $= 0.78$	$\dfrac{0.78}{0.775} = 1.01$
B/W varieties	$S_c = 30.89$	m-1=2	$\dfrac{S_c}{m-1} = \dfrac{30.89}{2}$ $= 15.445$	$\dfrac{15.445}{0.775} = 19.93$
Error (or) Residual	$S_d = 1.55$	(m-1)(m-2)	$\dfrac{S_d}{(m-1)(m-2)}$ $= \dfrac{1.55}{2}$ $= 0.775$	
Total	$S = 68.89$	$m^2 - 1 = 8$		

Critical value:

The value of 'F' for (2,8) d.f at 5% Los is 4.46

Conclusion:

Since F> the table value for the burners

$\therefore$ There is a significant difference between the burners

and also F> tabulated F for columns the difference b/w the engines is not significant.

Homework:

1. Analyse the variance in the following LS:

B 20	C 17	D 25	A 34
A 23	D 21	C 15	B 24
D 24	A 26	B 21	C 19
C 26	B 23	A 27	D 22

\

2. Analyse the variance in the following LS:

A 8	C 18	B 9
C 9	B 18	A 16
B 11	A 10	C 20

Factorial Experiments

Definition 1:

A factorial experiment in which each of m factors at 'S' is called a symmetrical factorial experiment and is often known as S^m factorial design

Definition 2:

2^m - Factorial experiments means a symmetrical factorial experiments where each of the m-factors is at two levels

2^2 -a factorial experiment means a symmetrical experiment where each of the factors is at two levels

Note:

If the numbers of level of the different factors are equal the experiments is called as a symmetrical factorial experiment.

Uses advantages of factorial experiments:

(i)   Factorial designs are widely used in experiments involving several factors where it is necessary
(ii)   F.D allow effects of a factor to be estimated at several levels of the others, giving conclusions that are valid over a range of experimental conditions
(iii)   The F.D are more efficient than one factor at a time experiments.
(iv)   In F.D individual factorial effect is estimated with precision, as whole of the experiment is devoted to it.
(v)   Factorial designs from the basis of other designs of considerable practical value.
(vi)   F.D are widely used in research work. These design are used to apply the results over a wide range of conditions

2^2 **-Factorial experiment:**

A factorial design with two factors, each at two levels is called a 2^2 factorial design

Yates's notation:

The two factors are denoted by the letters A and B the letters 'a' and 'b' denote one of the two levels of each of the corresponding factors and this will be called the second level.

The first level of A and B is generally expressed by the absence of the corresponding letter in the treatment combinations. The four treatment combinations can be enumerated as follows.

Symbols used:

a_0b_0(or)1:Factors A and B both at first level

a_1a_0(or)a:A at second level and B at first level

a_0a_1 (or) b :A at first level and B at second level

a_1a_1(or) ab : A and B both second levels.

Yates's method of computing factorial effect totals

For the calculation of various factorial effect total for 2^2-factorial experiments the following table is need

Treatment combination	Total yield from all replicates	(3)	(4)	Effect Totals
'1'	[1]	[1]+[a]	[1]+[a]+[b]+[ab]	Grand total
a	[a]	[b]+[ab]	[ab]-[b]+[a]-[1]	[A]
b	[b]	[a]-[1]	[ab]+[b]-[a]-[1]	[B]
ab	[ab]	[ab]-[b]	[ab]-[b]-[a]+[1]	[AB]

2^2-factorial experiment conducted in a CRD

Let $x_{ij} = $ jth observation of ith treatment combinations i=1, 2, 3, 4; j=1,2….(say)

i.e., $x_1 = [1]; x_2 = [a]; x_3 = [b]; x_4 = [ab]$

Where

x_i =total of ith treatment combination .

G=$\sum_i \sum_f x_{ij}$ grand total

n=4r=Total number of observations

$$TSS = \sum_j \sum_i x_{ij}^2 - \frac{G^2}{4r}$$

1. The follwoing table gives the plan and yields of a 2^2 – factorial experiment conducted in CRD

Analyse the design and give your comments

(1) 20	a 28	a 24	b 10
ab 23	b 11	ab 22	(1) 17
a 24	b 15	ab 21	(1) 19

Solution:

Arrange the observation as in one-way classification, we proceed as follows

Treatment Combination				Total
(1)	20	17	19	56
a	28	24	24	76
b	10	11	15	36
ab	23	22	21	66
Total			G=	234

Correction Formula $= \dfrac{G^2}{2^2 \times r} = \dfrac{234^2}{4 \times 3} = 4563$

$$\sum_j \sum_i x_{ij}^2 = 20^2 + 17^2 + 19^2 + 28^2 + 24^2 + 24^2 + 10^2 + 11^2 + 15^2 + 23^2 + 22^2 + 21^2$$

$$\sum_j \sum_i x_{ij}^2 = 4886$$

$$TSS = \sum_j \sum_i x_{ij}^2 - \frac{G^2}{4r} = 4886 - 4563 = 323$$

The values of SSA, SSB and SSAB are obtained by yate's method

Treatment combination	Total (2)	(3)	(4)	Divisor (5)	Sum of squares (6)
1	[1]	[1]+[a]	[1]+[a]+[b]+[ab]=[M]	-	-
a	[a]	[b]+[ab]	[ab]-[b]+[a]-[1]=[A]	4r	$[A]^2/4r$=SSA
b	[b]	[a]-[1]	[ab]+[b]-[a]-[1]=[B]	4r	$[B]^2/4r$=SSB
ab	[ab]	[ab]-[b]	[ab]-[b]-[a]+[1]=[AB]	4r	$[AB]^2/4r$=SSAB

SSE = TSS-(SSA+SSB+SSAB)

The analysis of variance table for 2^2 factorial design conducted in CRD

Source of variation	d.f	S.S	M.S.S	F
A	1	SSA	MSSA	$\dfrac{MSSA}{MSSE}$
B	1	SSB	MSSB	$\dfrac{MSSB}{MSSE}$
AB	1	SSAB	MSSAB	$\dfrac{MSSAB}{MSSE}$
Error	3(r-1)	SSE	MSSE	-
Total	4r-1	TSS	-	-

To obtain the sum of squares SSA, SSB, SSAB use yate's method:

Treatment/ combination (1)	Total response	(3)	(4)	Divisor (5)	S.S (6)
(1)	56	56+76=132	132+102=234	4r=12	Grand total
a	76	36+66=102	20+30=50	12	$\dfrac{50^2}{12}=208.33$
b	36	76-56=20	102-132=-30	12	$\dfrac{(-30)^2}{12}=75$
ab	66	66-36=30	30-20=10	12	$\dfrac{(10)^2}{12}=8.33$
				Total	291.66

SSE=TSS-(SSA+SSB+SSAB)

=323-291.66

SSE=31.34

Analysis of variance table:

Source of variation	d.f	S.S	M.S.S	F	$F_{0.01}(1,6)$
A	1	208.33	208.33	53.15	13.75
B	1	75	75	19.13	
AB	1	8.33	8.33	2.09	
Error	3(r-1)=6	31.34	3.92		
Total	4r-1=11	323			

Critical value:

The table value of for (1,6) d.f at 1% Los is 13.75

Conclusion:

Since F> tabulated value of 'F' for the main effect A and B, we conclude that the main effects A and B both are significantly different at 1% Los

3 Solution of Equations and Eigenvalue Problems

> Solution of algebraic and transcendental equations - Fixed point iteration method - Newton Raphson method- Solution of linear system of equations - Gauss elimination method - Pivoting - Gauss Jordan method - Iterative methods of Gauss Jacobi and Gauss Seidel - Matrix Inversion by Gauss Jordan method - Eigen values of a matrix by Power method.

3.1 Introduction

Solution of equation:

A value satisfies given equation is called solution of that equation.
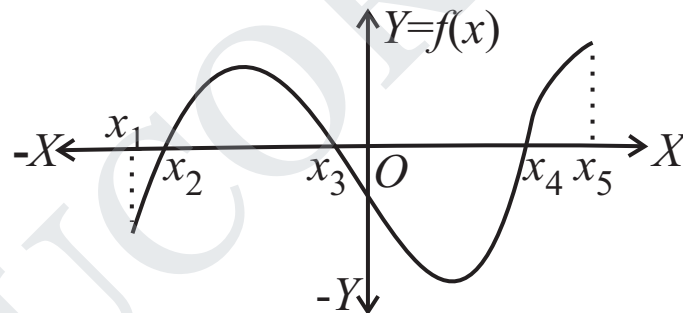
Here the curve $f(x)$ with

$$f(x = x_2) = f(x = x_3) = f(x = x_4) = 0$$

$$\Rightarrow x_2, x_3 \text{ and } x_4 \text{ are solutions of } f(x).$$

$$\text{and } f(x) > 0 \text{ for } x_2 < x < x_3 \text{ and } x_4 < x < x_5$$

$$f(x) < 0 \text{ for } x_1 < x < x_2 \text{ and } x_3 < x < x_4.$$

3.2 Solution of algebraic and transcendental equations

This chapter deals with finding solutions of algebraic and transcendental equations of either of the forms

$$f(x) = 0 \text{ or } f(x) = g(x) \tag{1}$$

where we want to solve for the unknown x. An algebraic equation is an equation constructed using the operations of $+, -, \times, \div$, and possibly root taking (radicals). Rational functions and polynomials are examples of algebraic functions. Transcendental equations in comparison are not algebraic. That is, they contain non-algebraic functions and possibly their inverses functions. Equations which contain either trigonometric functions, inverse trigonometric functions, exponential functions, and logarithmic functions are examples of non-algebraic functions which are called transcendental functions.

Example 1 : (Root of algebraic equation)

Estimate the solutions of the algebraic equation $f(x) = x^3 - \dfrac{132}{32}x^2 + \dfrac{28}{32}x + \dfrac{147}{32} = 0$.

Solution: We use a computer or calculator and plot a graph of the function $y = f(x)$ and obtain the figure as follows:
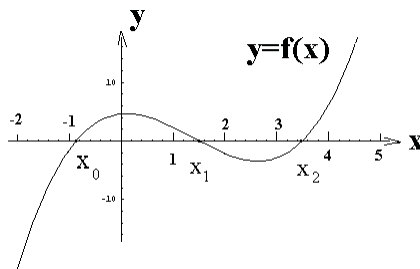
$$f(x) = x^3 - \frac{132}{32}x^2 + \frac{28}{32}x + \frac{147}{32} = 0$$

One can now estimate the solutions of the given equation by determining where the curve crosses the $x-$axis because these are the points where $y = 0$. Examining the graph in above figure we can place bounds on our estimates x_0, x_1, x_2 of the solutions. One such estimate is given by

$$-1.0 < x_0 < -0.8$$

$$1.4 < x_1 < 1.6$$

$$3.4 < x_2 < 3.6$$

To achieve a better estimate for the roots one can plot three versions of the above graph which have some appropriate scaling in the neighborhood of the roots.

Finding values for x where $f(x) = g(x)$ can also be approached using graphics. One can plot graphs of the curves $y = f(x)$ and $y = g(x)$ on the same set of axes and then try to estimate where these curves intersect.

Example 2 : (Root of transcendental equation)

Estimate the solutions of the transcendental equation $x^3 - \dfrac{132}{32}x^2 + \dfrac{28}{32}x + \dfrac{147}{32} = 5 \sin x$.

Solution: We again employ a computer or calculator and plot graphs of the functions

$$y = f(x) = x^3 - \frac{132}{32}x^2 + \frac{28}{32}x + \frac{147}{32} \text{ and } y = g(x) = 5 \sin x$$

to obtain the following figure
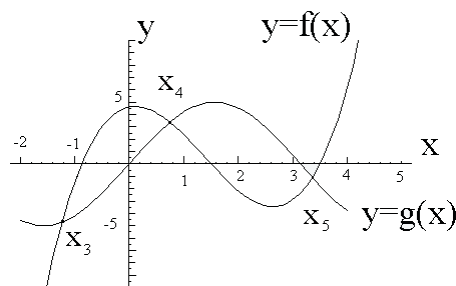
Graph of $y = f(x)$ and $y = g(x)$

One can estimate the points where the curve $y = f(x)$ intersects the curve $y = g(x)$. If the curves are plotted to scale on the same set of axes, then one can place bounds on the estimates of the solution. One such set of bounds is given by

$$-1.5 < x_3 < -1.0$$

$$0.5 < x_4 < 1.0$$

$$3.0 < x_5 < 3.5$$

By plotting these graphs over a ner scale one can obtain better estimates for the solutions.

Soln. of 1 equation	Soln. of n equations	Matrix Inversion	Eigen Val. & Vec.
Fixed point (IM)	Gauss elimination DM(RO)	Gauss Jordan(RO)	Power method IM(MM)
Newton Raphson (IM)	Gauss Jordan DM(RO)		
	Gauss Jacobi IM(SVS)		
	Gauss Seidel IM(LVS)		
Calculator Usage : Casio *fx*-991ms			
Polynomial:	**Simultaneous Equations**	**Matrix Entry** A:	**Matrix Entry** A&B:
EQN→Deg?(2 or 3)	EQN→Unknowns?(2 or 3)	Press[MODE]	Press[MODE]
Trigonometry:		[6:Matrix]	[6:Matrix]
Deg.,Rad.,Grads		1:Mat$A$	1:Mat$A$ & 2:Mat$B$
Exponential:		A^{-1}:	$A * B$:
Shift ln		Shift 4	Shift 4
Logarithmic:		MATRIX 3	MATRIX 3
log (def. base 10)		MATA x^{-1}	MATA * MATB

where **IM:**Iterative method, **DM:** Direct method, **RO:** Row operations, **MM:** Matrix multiplication.

   SVS: Step value substitution, **LVS:**Latest value substitution

   Deg.: Degree Mode, **Rad.:**Radian Mode, **def.:**default $\left(90°\text{degrees} = \frac{\pi}{2}\text{radians} = 100\text{Grads}\right)$

For example:

   $\sin 90°(\text{ in degree mode }) = \sin(\pi \div 2)(\text{ in radian mode }) = \sin(100)(\text{ in Grads mode }) = 1$

   *** $\sin(\pi \div 2) \neq \sin \pi \div 2$ [$\because \sin(\pi \div 2) = 1$ & $\sin \pi \div 2 = 0 \div 2 = 0$]

Definition of natural logarithm : When $e^y = x$. Then base e logarithm of x is : $ln(x) = log_e(x) = y$

Default base of log is 10 and ln is e.   [The e constant or Euler's number is: $e \approx 2.71828183$]

3.3 Fixed point iteration method

Write the given equation as $f(x) = 0$.

Find x_1, x_2 with opposite signs of $f(x)$. Find an relation as $x = g(x)$ such that

$$\left|g'(x_1)\right| < 1$$
$$\left|g'(x_2)\right| < 1$$

$$X_0 = \begin{cases} x_1, |f(x_1)| < |f(x_2)| \Rightarrow f(x_1) \text{ is nearer to zero.} \\ x_2, |f(x_1)| > |f(x_2)| \Rightarrow f(x_2) \text{ is nearer to zero.} \end{cases}$$

$$X_1 = g(X_0)$$

$$X_2 = g(X_1)$$

$$\vdots$$

$$X_n = g(X_{n-1})$$

$$X_{n+1} = g(X_n)$$

Note: Stop method if the consecutive values of $X_n \& X_{n+1}$ are equal upto required place of decimal. i.e., $X_n = X_n + 1$.

3.3.1 Part A

1. If $g(x)$ is continuous in $[a, b]$ then under what condition the iterative method $x = g(x)$ has unique solution in $[a, b]$.   (MJ2010)

   Solution: $|g'(x)| < 1$ in $[a, b]$ with order of convergence is one.

2. What are the advantages of iterative methods over direct methods for solving a system of linear equations.   (ND2012)

   Solution:

   1. Iterative methods are suitable for solving linear equations when the number of equations in a system is very large(more than 100).

   2. Iterative methods are very effective concerning computer storage and time requirements.

   3. One of the advantages of using iterative methods is that they require fewer multiplications for large systems.

   4. Iterative methods automatically adjust to errors during study. They can be implemented in smaller programmes than direct methods.

   5. They are fast and simple to use when coefficient matrix is sparse(more zeros).

   6. Advantageously they have fewer rounds off errors as compared to other direct methods.

   7. The direct methods, aim to calculate an exact solution in a finite number of operations. Whereas iterative methods begins with an initial approximation and reproduce usually improved approximations in an infinite sequence whose limit is the exact solution.

8. Error is controlled by the number of itertaions and there is no round of error.

9. Iterative methods are self-correcting.

*. Direct methods : Gaussian Elimination, Gauss Jordan Method

    Iteration Methods : Fixed-point Iteration, Gauss-Seidel Iteration Method

3. What do you mean by the order of convergence of an iterative method for finding the root of the equation $f(x) = 0$? (ND2013)

Solution: For Fixed point iteration method, $\left|g'(x)\right| < 1$ in $[a, b]$, with order of convergence is one.

$$\text{where } f(x) = 0 \Rightarrow x = g(x)$$

For Newton-Raphson iteration method, $|f(x)f''(x)| < |f'(x)|^2$, with order of convergence is two.

$$\boxed{\textbf{Worked Examples}}$$

3.3.2 Part B

Example 3.1. Solve the equation $x^3 + x^2 - 100 = 0$ by iteration method.

Solution: Let $f(x) = x^3 + x^2 - 100 = 0$

$$f(x = 0) = (0)^3 + (0)^2 - 100 = -100 < 0 \ ('-'ve)$$
$$f(x = 1) = (1)^3 + (1)^2 - 100 = -98 < 0 \ ('-'ve)$$
$$f(x = 2) = (2)^3 + (2)^2 - 100 = -88 < 0 \ ('-'ve)$$
$$f(x = 3) = (3)^3 + (3)^2 - 100 = -64 < 0 \ ('-'ve)$$
$$\boxed{f(x = 4) = (4)^3 + (4)^2 - 100 = -20 < 0 \ ('-'ve)}$$
$$\boxed{f(x = 5) = (5)^3 + (5)^2 - 100 = 50 > 0 \ ('+'ve)}$$

$\therefore f(x)$ has a root between 4 and 5.

i.e, $f(x)$ has a positive root.

This equation $f(x) = x^3 + x^2 - 100 = 0$ can be written as

$$x^3 + x^2 - 100 = 0$$
$$i.e., x^2(x + 1) = 100$$
$$i.e., x = \frac{10}{\sqrt{x + 1}} \ [= g(x), \text{ say}]$$
$$\boxed{\therefore g(x) = \frac{10}{\sqrt{x + 1}} = 10(x + 1)^{\frac{-1}{2}}}$$

Therefore, $g'(x) = \dfrac{d}{dx}[g(x)]$

$$= \dfrac{d}{dx}\left[10(x+1)^{\frac{-1}{2}}\right] = 10\left(\dfrac{-1}{2}\right)(x+1)^{\frac{-1}{2}-1} = -5(x+1)^{\frac{-3}{2}} = \dfrac{-5}{(x+1)^{\frac{3}{2}}}$$

$$\therefore \left|g'(x)\right| = \left|\dfrac{-5}{(x+1)^{\frac{3}{2}}}\right| = \dfrac{5}{(x+1)^{\frac{3}{2}}}$$

At $x = 4, \left|g'(x=4)\right| = \dfrac{5}{(4+1)^{3/2}} = \dfrac{5}{(5)^{3/2}} = \dfrac{5}{5\sqrt{5}} = \dfrac{1}{\sqrt{5}} = \dfrac{1}{2.23606} = 0.57735 < 1$

At $x = 5, \left|g'(x=5)\right| = \dfrac{5}{(5+1)^{3/2}} = \dfrac{5}{(6)^{3/2}} = 0.34020 < 1$

$$\boxed{|g'(x)| < 1 \text{ in } (4,5)}$$

So the fixed point iteration method can be applied. Let $x_0 = 4$ $\qquad\qquad [|f(x=4)| < |f(x=5)|]$

Use $x_{n+1} = g(x_n)$

$$x_1 = g(x_0) = \dfrac{10}{\sqrt{x_0+1}} = \dfrac{10}{\sqrt{4+1}} = \dfrac{10}{\sqrt{5}} = 4.47214 \qquad \text{(round off at } 5^{th} \text{ decimal)}$$

$$x_2 = g(x_1) = \dfrac{10}{\sqrt{4.47214+1}} = 4.27486$$

$$x_3 = g(x_2) = \dfrac{10}{\sqrt{4.27486+1}} = 4.35406$$

$$x_4 = g(x_3) = \dfrac{10}{\sqrt{4.35406+1}} = 4.32174$$

$$x_5 = g(x_4) = \dfrac{10}{\sqrt{4.32174+1}} = 4.33484$$

$$x_6 = g(x_5) = \dfrac{10}{\sqrt{4.33484+1}} = 4.32952$$

$$x_7 = g(x_6) = \dfrac{10}{\sqrt{4.32952+1}} = 4.33168$$

$$x_8 = g(x_7) = \dfrac{10}{\sqrt{4.33168+1}} = 4.33080$$

$$x_9 = g(x_8) = \dfrac{10}{\sqrt{4.33080+1}} = 4.33116$$

$$x_{10} = g(x_9) = \dfrac{10}{\sqrt{4.33116+1}} = 4.33101$$

$$x_{11} = g(x_{10}) = \dfrac{10}{\sqrt{4.33101+1}} = 4.33107$$

$$\boxed{x_{12} = g(x_{11}) = \dfrac{10}{\sqrt{4.33107+1}} = 4.33105}$$

$$\boxed{x_{13} = g(x_{12}) = \dfrac{10}{\sqrt{4.33105+1}} = 4.33105}$$

These values are same at fifth decimal, i.e., $x_{12} = x_{13}$. So stop the method.

$$\boxed{\therefore \text{The root of the given equation } x^3 + x^2 - 100 = 0 \text{ is } 4.33105}$$

Note: In above problem, the root is 4.33105(at 5th decimal) or 4.3311(at 4th decimal) or 4.331(at 3rd decimal).

Example 3.2. Solve the equation $x^3 + 2x^2 + 10x = 20$ by fixed point iteration method.
Solution: $[x_{13} = x_{14} = 1.36880]$

Example 3.3. Solve the equation $x^3 + x + 1 = 0$ by fixed point iteration method.
Solution: $\left[g(x) = (-1-x)^{1/3}\,;\, x_{23} = x_{24} = -0.682\right]$

Example 3.4. Solve the equation $3x - \cos x - 2 = 0$ by fixed point iteration method correct to 3 decimal places.
Solution: Let $f(x) = 3x - \cos x - 2 = 0$

$$\boxed{f(x=0) = 3(0) - \cos(0) - 2 = 0 - 1 - 2 = -3 < 0\,('-'\,ve)}$$

$$\boxed{f(x=1) = 3(1) - \cos(1) - 2 = 0.459697 > 0\,('+'\,ve)}$$

$\therefore f(x)$ has a root between 0 and 1.

i.e, $f(x)$ has a positive root.

This equation $f(x) = 3x - \cos x - 2 = 0$ can be written as

$$3x - \cos x - 2 = 0$$
$$\text{i.e., } 3x = 2 + \cos x$$
$$\text{i.e., } x = \frac{2 + \cos x}{3}\ [= g(x),\ \text{say}]$$
$$\boxed{\therefore g(x) = \frac{2 + \cos x}{3}}$$

Therefore, $g'(x) = \dfrac{d}{dx}[g(x)]$

$$= \frac{d}{dx}\left[\frac{2 + \cos x}{3}\right] = \left(\frac{1}{3}\right)\frac{d}{dx}[2 + \cos x] = \left(\frac{1}{3}\right)[0 - \sin x] = \frac{-\sin x}{3}$$

$$\therefore \left|g'(x)\right| = \left|\frac{-\sin x}{3}\right| = \frac{\sin x}{3}$$

At $x = 0, \left|g'(x=0)\right| = \dfrac{\sin(0)}{3} = 0 < 1$

At $x = 1, \left|g'(x=1)\right| = \dfrac{\sin(1)}{3} = 0.2804 < 1$

$$\boxed{\left|g'(x)\right| < 1 \text{ in } (0, 1)}$$

So the fixed point iteration method can be applied. Let $x_0 = 0$ $[|f(x=1)| < |f(x=0)|]$

Use $x_{n+1} = g(x_n)$ Now, put $n = 0$ in $x_{n+1} = g(x_n)$, we get

$$x_1 = g(x_0) = \frac{2 + \cos(x_0)}{3} = \frac{2 + \cos(0)}{3} = \frac{2 + 1}{3} = 1$$

$$x_2 = g(x_1) = \frac{2 + \cos(x_1)}{3} = \frac{2 + \cos(1)}{3} = 0.84677$$

$$x_3 = g(x_2) = \frac{2 + \cos(x_2)}{3} = \frac{2 + \cos(0.84677)}{3} = 0.88747$$

$$x_4 = g(x_3) = \frac{2 + \cos(x_3)}{3} = \frac{2 + \cos(0.84747)}{3} = 0.87713$$

$$\boxed{x_5 = g(x_4) = \frac{2 + \cos(x_4)}{3} = \frac{2 + \cos(0.87713)}{3} = 0.87979}$$

$$\boxed{x_6 = g(x_5) = \frac{2 + \cos(x_5)}{3} = \frac{2 + \cos(0.87979)}{3} = 0.87910}$$

These values are same at third decimal, i.e., $x_5 = x_6$. So stop the method.

$$\boxed{\therefore \text{ The root of the given equation } 3x - \cos x - 2 = 0 \text{ is } 0.879}$$

Example 3.5. **Solve the equation $\cos x = 3x - 1$ by iteration method upto 5 decimal.**

(Or) Find a positive root of the equation $\cos x - 3x + 1 = 0$ by using iteration method. (AM13)

Solution: $[x = 0.60710]$

Example 3.6. **Solve the equation $2\sin x = x$ by fixed point iteration method.**

Solution: $[x = 1.89549]$

$$\{\text{Hint: } |g'(x = 1)| > 1, \text{ so use } |g'(x = 1.5)| < 1 \,\&\, |g'(x = 2)| < 1\}$$

Example 3.7. **Solve the equation $e^x - 3x = 0$ by iteration method correct to 3 decimal.(MJ12)**

Solution: Let $f(x) = e^x - 3x = 0$

$$\boxed{\begin{aligned} f(x = 0) &= e^0 - 3(0) = 1 > 0 \;('+'\,ve) \\ f(x = 1) &= e^1 - 3(1) = -0.2817 < 0 \;('-'\,ve) \end{aligned}}$$

$\therefore f(x)$ has a root between 0 and 1.

i.e, $f(x)$ has a positive root.

This equation $f(x) = e^x - 3x = 0$ can be written as

$$3x = e^x$$

$$i.e., x = \frac{e^x}{3} \;[= g(x), \text{ say}]$$

$$\boxed{\therefore g(x) = \frac{e^x}{3}}$$

Therefore, $g'(x) = \dfrac{d}{dx}[g(x)] = \dfrac{d}{dx}\left[\dfrac{e^x}{3}\right] = \dfrac{e^x}{3}$

$$\therefore |g'(x)| = \left|\frac{e^x}{3}\right| = \frac{e^x}{3}$$

At $x = 0, |g'(x = 0)| = \dfrac{e^0}{3} = \dfrac{1}{3} = 0.333 < 1$

At $x = 1, |g'(x = 1)| = \dfrac{e^1}{3} = 0.906 < 1$

$$\boxed{|g\,'(x)| < 1 \text{ in } (0, 1)}$$

So the fixed point iteration method can be applied. Let $x_0 = 0$ use $x_{n+1} = g(x_n)$

Now, put $n = 0$ in $x_{n+1} = g(x_n)$, we get

$x_1 = g(x_0) = \dfrac{e^{x_0}}{3} = \dfrac{e^0}{3} = 0.33333$	$x_8 = \dfrac{e^{x_7}}{3} = \dfrac{e^{0.60727}}{3} = 0.61180$
$x_2 = \dfrac{e^{x_1}}{3} = \dfrac{e^{0.33333}}{3} = 0.46520$	$x_9 = \dfrac{e^{x_8}}{3} = \dfrac{e^{0.61180}}{3} = 0.61458$
$x_3 = \dfrac{e^{x_2}}{3} = \dfrac{e^{0.46520}}{3} = 0.53078$	$x_{10} = \dfrac{e^{x_9}}{3} = \dfrac{e^{0.61458}}{3} = 0.61629$
$x_4 = \dfrac{e^{x_3}}{3} = \dfrac{e^{0.53078}}{3} = 0.56675$	$x_{11} = \dfrac{e^{x_{10}}}{3} = \dfrac{e^{0.61629}}{3} = 0.61735$
$x_5 = \dfrac{e^{x_4}}{3} = \dfrac{e^{0.56675}}{3} = 0.58751$	$\boxed{x_{12} = g(x_{11}) = \dfrac{e^{x_{11}}}{3} = \dfrac{e^{0.61735}}{3} = 0.61800}$
$x_6 = \dfrac{e^{x_5}}{3} = \dfrac{e^{0.58751}}{3} = 0.59983$	$\boxed{x_{13} = g(x_{12}) = \dfrac{e^{x_{12}}}{3} = \dfrac{e^{0.61800}}{3} = 0.61840}$
$x_7 = \dfrac{e^{x_6}}{3} = \dfrac{e^{0.59983}}{3} = 0.60727$	

These values are same at third decimal, i.e., $x_{12} = x_{13}$. So stop the method.

$$\boxed{\therefore \text{ The root of the given equation } e^x - 3x = 0 \text{ is } 0.618}$$

Example 3.8. Solve the equation $2x - \log x = 7$ by iteration method correct to 5 decimal.

Solution: Let $f(x) = 2x - \log x = 7 = 0$

$$f(x = 0) = 2(0) - \log_{10}(0) - 7 = \text{(Undefined)}$$

$$f(x = 1) = 2(1) - \log_{10}(1) - 7 = -5 < 0 \ ('-' ve)$$

$$f(x = 2) = 2(2) - \log_{10}(2) - 7 = -3 < 0 \ ('-' ve)$$

$$f(x = 3) = 2(3) - \log_{10}(3) - 7 = -1 < 0 \ ('-' ve)$$

$$f(x = 4) = 2(4) - \log_{10}(4) - 7 = 0.39 > 0 \ ('+' ve)$$

$\therefore f(x)$ has a root between 3 and 4.

i.e, $f(x)$ has a positive root.

This equation $f(x) = 2x - \log x - 7 = 0$ can be written as

$$2x = 7 + \log x$$

$$\text{i.e., } x = \frac{7 + \log x}{2} \ [= g(x), \text{ say}]$$

$$\boxed{\therefore g(x) = \frac{7 + \log x}{2}}$$

Therefore, $g\,'(x) = \dfrac{d}{dx}[g(x)] = \dfrac{d}{dx}\left[\dfrac{7 + \log x}{2}\right] = \dfrac{\frac{1}{x}}{2} = \dfrac{1}{2x}$

$$\therefore |g\,'(x)| = \left|\frac{1}{2x}\right| = \frac{1}{2x}$$

$$\text{At } x = 3, |g\,'(x = 3)| = \frac{1}{2(3)} = \frac{1}{6} = 0.166 < 1$$

$$\text{At } x = 4, |g\,'(x = 4)| = \frac{1}{2(4)} = \frac{1}{8} = 0.125 < 1$$

$$\boxed{|g'(x)| < 1 \text{ in } (3, 4)}$$

So the fixed point iteration method can be applied. Let $x_0 = 4$, use $x_{n+1} = g(x_n)$

Now, put $n = 0$ in $x_{n+1} = g(x_n)$, we get

$$x_1 = g(x_0) = \frac{7 + \log(x_0)}{2} = \frac{7 + \log(4)}{2} = 3.80103$$

$$x_2 = g(x_1) = \frac{7 + \log(x_1)}{2} = \frac{7 + \log(3.80103)}{2} = 3.78995$$

$$x_3 = g(x_2) = \frac{7 + \log(x_2)}{2} = \frac{7 + \log(3.78995)}{2} = 3.78932$$

$$x_4 = g(x_3) = \frac{7 + \log(x_3)}{2} = \frac{7 + \log(3.78932)}{2} = 3.78928$$

$$x_5 = g(x_4) = \frac{7 + \log(x_4)}{2} = \frac{7 + \log(3.78928)}{2} = 3.78928$$

These values are same at fifth decimal, i.e., $x_4 = x_5$. So stop the method.

$$\boxed{\therefore \text{ The root of the given equation } 2x - \log x = 7 \text{ is } 3.78928}$$

Example 3.9. **Solve the equation $3x - \log_{10} x = 6$ by fixed point iteration method.**

Solution: $\hspace{9cm} [x = 2.1080]$

3.3.3 Anna University Questions

1. Solve $e^x - 3x = 0$ by the method of fixed point iteration. $\hspace{1.5cm}$ (AM12)

    Solution : $\hspace{9cm} [x = 0.618]$

2. Find a positive root of the equation $cos x - 3x + 1 = 0$ by using iteration method. $\hspace{0.5cm}$ (AM13)

    Solution : $\hspace{8.5cm} [x = 0.60710]$

3.4 Newton Raphson method

(Newton's method or method of tangents)

Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = \phi(x_n)$, where $n = 0, 1, 2, \cdots$

3.4.1 Part A

1. State the order of convergence and convergence condition for Newton-Raphson method.

    (Or) What is the criterion for the convergence in Newton's method? $\hspace{1cm}$ (MJ2011)

    (Or) State the order of convergence and the criterion for the convergence in Newton's method.

    $\hspace{11cm}$ (AM2012)

    (Or) Write down the condition for convergence of Newton-Raphson method for $f(x) = 0$. $\hspace{0.3cm}$ (ND14)

    Solution: Order of convergence is 2. Convergence condition is

    $$|f(x) f''(x)| < |f'(x)|^2$$

2. Find an iterative formula to find $\sqrt{N}$, where N is positive number.

Solution: Let $x = \sqrt{N}$

$$\Rightarrow x^2 = N \Rightarrow x^2 - N = 0$$

Let $f(x) = x^2 - N \Rightarrow f'(x) = 2x$

By Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{2x_n^2 - x_n^2 + N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

3. Derive Newton's algorithm for finding the pth root of a number N.

Solution: Let $x = N^{1/p} = \sqrt[p]{N} \Rightarrow x^p - N = 0$

Let $f(x) = x^p - N \Rightarrow f'(x) = px^{p-1}$

By Newton's formula

$$x_{n+1} = x_n - \frac{x_n^p - N}{px^{p-1}} = \frac{px_n^p - x_n^p + N}{px^{p-1}} = \frac{(p-1)x_n^p + N}{px^{p-1}}$$

4. Establish an iteration formula to find the reciprocal of a positive number N by Newton-Raphson method. (Or)

   Find an iterative formula to find the reciprocal of a given number $N(N \neq 0)$. (AM2013)

Solution : Let $x = \dfrac{1}{N}$

i.e., $N = \dfrac{1}{x}$

Let $f(x) = \dfrac{1}{x} - N$

$f'(x) = -\dfrac{1}{x^2}$

We know that by Newton's iterative formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \left[\dfrac{\dfrac{1}{x_n} - N}{-\dfrac{1}{x_n^2}} \right]$

$$= x_n + x_n^2 \left[\frac{1}{x_n} - N \right] = x_n + x_n - Nx_n^2 = 2x_n - Nx_n^2$$

$$x_{n+1} = x_n [2 - Nx_n]$$

which is the required iterative formula for the reciprocal of a given number N.

5. Locate the negative root of $x^3 - 2x + 5 = 0$, approximately.

Solution: Let $f(x) = x^3 - 2x + 5 = 0$

$$f(-1) = -1 + 2 + 5 = 6 \, (+ve)$$

$$f(-2) = -8 + 4 + 5 = 1 \, (+ve)$$

$$f(-3) = -27 + 6 + 5 = -16 \, (-ve)$$

$\therefore$ Root lies between -2 and -3.

and root is closer to -2 since $|f(-2)| < |f(-3)|$

6. Evaluate $\sqrt{12}$ applying Newton formula.

   Solution: Let $x = \sqrt{12}$

   $$\Rightarrow x^2 = 12 \Rightarrow x^2 - 12 = 0$$

   Let $f(x) = x^2 - 12 \Rightarrow f'(x) = 2x$

   $\quad f(3) = 9 - 12 = -3\,(-ve)$

   $\quad f(4) = 16 - 12 = 4\,(+ve)$

   $\therefore$ Root lies between 3 and 4.

   and root is closer to 3 since $|f(3)| < |f(4)|$

   Take $x_0 = 3$

   Newton's formula is

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

   when $n = 0, x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 3 - \dfrac{f(3)}{f'(3)} = 3 - \dfrac{(3)^2 - 12}{2(3)} = 3.5$

   when $n = 1, x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 3.5 - \dfrac{f(3.5)}{f'(3.5)} = 3.5 - \dfrac{(3.5)^2 - 12}{2(3.5)} = 3.46429$

   when $n = 2, x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 3.46429 - \dfrac{f(3.46429)}{f'(3.46429)} = 3.46429 - \dfrac{(3.46429)^2 - 12}{2(3.46429)} = 3.46410$

   when $n = 3, x_4 = x_3 - \dfrac{f(x_3)}{f'(x_3)} = 3.46410 - \dfrac{f(3.46410)}{f'(3.46410)} = 3.46410 - \dfrac{(3.46410)^2 - 12}{2(3.46410)} = 3.46410$

   Here $x_3 = x_4 = 3.46410$

   $\therefore$ The root is 3.46410

7. What is Newton's algorithm to solve the equation $x^2 = 12$?   (ND2010)

   Solution : Let $f(x) = x^2 - 12 \Rightarrow f'(x) = 2x$

   We know that by

   Newton Raphson iteration formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

   $$= x_n - \frac{x_n^2 - 12}{2x_n}$$

   $$= \frac{2x_n^2 - x_n^2 + 12}{2x_n}$$

   $$x_{n+1} = \frac{x_n^2 + 12}{2x_n}$$

8. Using Newton's method, find the root between 0 and 1 of $x^3 = 6x - 4$.   (ND2011)

   Solution: Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $n = 0, 1, 2, \cdots$ (1)

   Let $f(x) = x^3 - 6x + 4 \Rightarrow f'(x) = 3x^2 - 6$

   $f(x = 0) = 0^3 - 6(0) + 4 = 4 > 0 \ ('+' \text{ve})$

   $f(x = 1) = 1^3 - 6(1) + 4 = -1 < 0 \ ('-' \text{ve})$

$\therefore$ A root lies between 0 and 1.

But $|f(0)| > |f(1)|$

$\therefore$ The root is nearer to 1.

Take $x_0 = 1$

Now, put $n = 0$ in (1), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{(x_0)^3 - 6(x_0) + 4}{\left[3(x_0)^2 - 6\right]} = 1 - \frac{\left[1^3 - 6(1) + 4\right]}{\left[3(1)^2 - 6\right]}$$

$$= 1 - \frac{(-1)}{-3} = 1 - \frac{1}{3} = 0.666 = 0.67 \qquad \text{[Corrected to 2 dec. places]}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1)^3 - 6(x_1) + 4}{\left[3(x_1)^2 - 6\right]} = 0.67 - \frac{\left[(0.67)^3 - 6(0.67) + 4\right]}{\left[3(0.67)^2 - 6\right]}$$

$$= 0.67 - \frac{0.28}{-4.65} = 0.67 + \frac{0.28}{4.68} = 0.73$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{(x_2)^3 - 6(x_2) + 4}{\left[3(x_2)^2 - 6\right]} = 0.73 - \frac{\left[(0.73)^3 - 6(0.73) + 4\right]}{\left[3(0.73)^2 - 6\right]}$$

$$= 0.73 - \frac{0.009}{-4.4013} = 0.73 + \frac{0.009}{4.4013} = 0.7320 = 0.73 \qquad \text{[Corrected to 2 dec. places]}$$

Here $x_2 = x_3 = 0.73$

$\therefore$ A root of given equation is 0.73 corrected to two decimal places.

9. Evaluate $\sqrt{15}$ using Newton-Raphson's formula.   (AM2014)

   Solution : W.K.T. The Newton Raphson iterative formula for $\sqrt{N}$ is

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$$

To find $\sqrt{15}$

W.K.T. $\sqrt{15}$ is nearer to 4

$\therefore$ Let $x_0 = 4$

$$x_1 = \frac{1}{2}\left[x_0 + \frac{N}{x_0}\right] = \frac{1}{2}\left[4 + \frac{15}{4}\right] = 3.87500$$

$$x_2 = \frac{1}{2}\left[x_1 + \frac{N}{x_1}\right] = \frac{1}{2}\left[3.875 + \frac{15}{3.875}\right] = 3.87298$$

$$x_3 = \frac{1}{2}\left[x_2 + \frac{N}{x_2}\right] = \frac{1}{2}\left[3.87298 + \frac{15}{3.87298}\right] = 3.87298$$

Here $x_2 = x_3 = 3.87298$

$\therefore$ The root is 3.87298

3.4.2 Part B

Example 3.10.   Using Newton-Raphson method, find a ' + 've root correct to 3 decimal places for the equation $x^3 - x - 2 = 0$.

Solution: Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, where $n = 0, 1, 2, \cdots$

Let $f(x) = x^3 - x - 2$

$\Rightarrow f'(x) = 3x^2 - 1$

$f(x = 0) = 0^3 - 0 - 2 = -2 < 0 \; ('-'ve)$

$f(x = 1) = 1^3 - 1 - 2 = -2 < 0 \; ('-'ve)$

$f(x = 2) = 2^3 - 2 - 2 = 4 > 0 \; ('+'ve)$

$\therefore$ A root lies between 1 and 2.

Take $x_0 = 1$

Now, put $n = 0$ in $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{(x_0)^3 - x_0 - 2}{\left[3(x_0)^2 - 1\right]} = 1 - \frac{\left[1^3 - 1 - 2\right]}{\left[3(1)^2 - 1\right]} = 2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1)^3 - x_1 - 2}{\left[3(x_1)^2 - 1\right]} = 2 - \frac{\left[2^3 - 2 - 2\right]}{\left[3(2)^2 - 1\right]} = 1.63636$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{(x_2)^3 - x_2 - 2}{\left[3(x_2)^2 - 1\right]} = 1.63636 - \frac{1.63636^3 - 1.63636 - 2}{\left[3(1.63636)^2 - 1\right]} = 1.53039$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_3 - \frac{(x_3)^3 - x_3 - 2}{\left[3(x_3)^2 - 1\right]} = 1.53039 - \frac{1.53039^3 - 1.53039 - 2}{\left[3(1.53039)^2 - 1\right]} = 1.52144$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = x_4 - \frac{(x_4)^3 - x_4 - 2}{\left[3(x_4)^2 - 1\right]} = 1.52144 - \frac{1.52144^3 - 1.52144 - 2}{\left[3(1.52144)^2 - 1\right]} = 1.52138$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = x_5 - \frac{(x_5)^3 - x_5 - 2}{\left[3(x_5)^2 - 1\right]} = 1.52138 - \frac{1.52138^3 - 1.52138 - 2}{\left[3(1.52138)^2 - 1\right]} = 1.52138$$

$x_5 = x_6 = 1.52138$ upto 5 decimal places.

$\therefore$ A root of given equation is 1.521

Example 3.11. Find the real root of $x^3 - 3x + 1 = 0$ lying between 1 and 2 by Newton Raphson method.

Solution: Let $f(x) = x^3 - 3x + 1$

$f(1) = 1 - 3 + 1 = -1 < 0$

$f(2) = 8 - 6 + 1 = 3 > 0$

$\therefore$ A root lies between 1 and 2.

Since $|f(1)| < |f(2)|$, the root is nearer to 1.

Let $x_0 = 1$.

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = x^3 - 3x + 1 \Rightarrow f'(x) = 3x^2 - 3$$

Since $f'(1) = 3 - 30$, Newton's approximation formula cannot be applied for the initial approximation at $x = 1$.

Let us take $x_0 = 1.5$ Now,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1.5 - \frac{f(1.5)}{f'(1.5)}$$

$$= 1.5 - \left[\frac{(1.5)^3 - 3(1.5) + 1}{3(1.5)^2 - 3}\right] = 1.5 - \left[\frac{-0.125}{3.75}\right]$$

$$= 1.5 + 0.0333$$

$$x_1 = 1.5333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.5333 - \frac{f(1.5333)}{f'(1.5333)}$$

$$= 1.5333 - \left[\frac{(1.5333)^3 - 3(1.5333) + 1}{3(1.5333)^2 - 3}\right]$$

$$= 1.5333 - \left[\frac{0.0049}{4.0530}\right] = 1.5333 - 0.0012$$

$$x_2 = 1.5321$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.5321 - \frac{f(1.5321)}{f'(1.5321)}$$

$$= 1.5321 - \left[\frac{(1.5321)^3 - 3(1.5321) + 1}{3(1.5321)^2 - 3}\right]$$

$$= 1.5321 - \left[\frac{0.00004}{4.042}\right] = 1.5321 - 0.00001$$

$$x_3 = 1.5321$$

Here $x_2 = 1.5321$ and

$x_3 = 1.5321$.

∴**The required root,**

correct to four places of decimal is

1.5321.

Example 3.12.   Solve the equation $x^3 - 5x + 3 = 0$ by Newton-Raphson method.

Solution: $[x_3 = x_4 = 0.65662]$

Example 3.13.   Solve the equation $x^4 - x - 9 = 0$ by Newton-Raphson method.

Solution: $[x_6 = x_7 = 1.813]$ upto 3rd decimal.

Example 3.14. Find a negative root of $x^3 - 2x + 5 = 0$ by Newton Raphson method.

Solution: (Home work)

[Hint: $f(-2) > 0, f(-3) < 0$

∴ a root lies between -2 and -3

Ans : -2.095]

[Or]

$\Big[$Given: $x^3 - 2x + 5 = 0$

Replace x by $-x^3 + 2x + 5 = 0$

i.e., $x^3 - 2x - 5 = 0$

Let $f(x) = x^3 - 2x - 5 = 0$

Now, we find the positive root of $f(x) = x^3 - 2x - 5$

$f(2) = -1$ and $f(3) = 16$

∴ a root lies between 2 and 3

The root of $f(x) = x^3 - 2x - 5$ is 2.095

∴ The root of $x^3 - 2x + 5$ is $-2.095\Big]$

Example 3.15.   Using Newton-Raphson method, find a ' + 've root correct to 5 decimal places for the equation $x \tan x = 1.28$

Solution: Newton-Raphson method formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where } n = 0, 1, 2, \cdots$$

Let $f(x) = x \tan x - 1.28 \Rightarrow f'(x) = x \sec^2 x + \tan x$

$$f(x = 0) = (0) \tan(0) - 1.28 = -1.28 < 0 \;('-'\text{ve})$$

$$f(x = 0.5) = (0.5) \tan(0.5) - 1.28 = -1.006 < 0 \;('-'\text{ve})$$

$$f(x = 1) = (1) \tan(1) - 1.28 = 0.2774 > 0 \;('+'\text{ve})$$

$\therefore$ A root lies between 0.5 and 1.

Take $x_0 = 0.5$

Now, put $n = 0$ in $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0 \tan x_0 - 1.28}{x_0 \sec^2 x_0 + \tan x_0} = 0.5 - \frac{(0.5)\tan(0.5) - 1.28}{(0.5)\sec^2(0.5) + \tan(0.5)} = 1.34218$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.34218 - \frac{(1.34218)\tan(1.34218) - 1.28}{(1.34218)\sec^2(1.34218) + \tan(1.34218)} = 1.19469$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.19469 - \frac{(1.19469)\tan(1.19469) - 1.28}{(1.19469)\sec^2(1.19469) + \tan(1.19469)} = 1.04143$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.04143 - \frac{(1.04143)\tan(1.04143) - 1.28}{(1.04143)\sec^2(1.04143) + \tan(1.04143)} = 0.95512$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.95512 - \frac{(0.95512)\tan(0.95512) - 1.28}{(0.95512)\sec^2(0.95512) + \tan(0.95512)} = 0.93871$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 0.93871 - \frac{(0.93871)\tan(0.93871) - 1.28}{(0.93871)\sec^2(0.93871) + \tan(0.93871)} = 0.93826$$

$$x_7 = x_6 - \frac{f(x_6)}{f'(x_6)} = 0.93826 - \frac{(0.93826)\tan(0.93826) - 1.28}{(0.93826)\sec^2(0.93826) + \tan(0.93826)} = 0.93826$$

$x_6 = x_7 = 0.93826$ upto 5 decimal places.

$\therefore$ A root of given equation is 0.93826

Example 3.16. **Solve the equation $x \sin x + \cos x = 0$ by Newton-Raphson method.**

Solution: $[x_4 = x_5 = 2.79839]$

Example 3.17. **Using Newton-Raphson method, find a root correct to 5 decimal places for the equation $e^x - 4x = 0$**

Solution: Newton-Raphson method formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Let $f(x) = e^x - 4x \Rightarrow f'(x) = e^x - 4$

$$f(x = 0) = e^0 - 4(0) = 1 > 0 \;('+'\text{ ve})$$

$$f(x = 1) = e^1 - 4(1) = -1.28272 < 0 \;('-'\text{ve})$$

$\therefore$ A root lies between 0 and 1.

Here $|f(1)| = 1.28172 > |f(0)| = 1$

$\therefore$ Root is nearer to 0.

Take $x_0 = 0$

Now, put $n = 0$ in $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{e^{x_0} - 4x_0}{e^{x_0} - 4} = 0 - \frac{e^0 - 4(0)}{e^0 - 4} = 0.33333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{e^{x_1} - 4x_1}{e^{x_1} - 4} = 0.33333 - \frac{e^{0.33333} - 4(0.33333)}{e^{0.33333} - 4} = 0.35725$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{e^{x_2} - 4x_2}{e^{x_2} - 4} = 0.35725 - \frac{e^{0.35725} - 4(0.35725)}{e^{0.35725} - 4} = 0.35740$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_3 - \frac{e^{x_3} - 4x_3}{e^{x_3} - 4} = 0.35740 - \frac{e^{0.35740} - 4(0.35740)}{e^{0.35740} - 4} = 0.35740$$

$\therefore$ A root of given equation is 0.35740

Example 3.18. Find the real root of $xe^x - 2 = 0$ correct to three places of decimals using Newton Raphson method.

Solution: Let $f(x) = xe^x - 2$

$$f(0) = -2 < 0$$
$$f(1) = e - 2 = 0.7183 > 0$$

Since $f(0) < 0$ and $f(1) > 0$.

$\therefore$ A root lies between 0 and 1.

Since $|f(1)| < |f(0)|$, the root is nearer to 1.

Let $x_0 = 1$.

The Newton Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$f(x) = xe^x - 2 \Rightarrow f'(x) = xe^x - e^x$$

Now,

$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \frac{f(1)}{f'(1)} \\ &= 1 - \left[\frac{e-2}{e+e}\right] \\ &= 1 - \left[\frac{0.7183}{5.4366}\right] \\ &= 1 - 01321 \\ x_1 &= 0.8679 \end{aligned}$	$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 0.8679 - \frac{(0.8679)e^{0.8679} - 2}{(0.8679)e^{0.8679} + e^{0.8679}} \\ &= 0.8679 - \left[\frac{0.0673}{4.4492}\right] \\ &= 0.8679 - 0.0151 \\ x_2 &= 0.8528 \end{aligned}$	$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 0.8528 - \frac{(0.8528)e^{0.8528} - 2}{(0.8528)e^{0.8528} + e^{0.8528}} \\ &= 0.8528 - \left[\frac{0.0008}{4.3471}\right] \\ &= 0.8528 - 0.0002 \\ x_3 &= 0.8526 \end{aligned}$

Here $x_2 = 0.853$ and $x_3 = 0.853$.

$\therefore$ The required root, correct to three places of decimal is 0.853.

Example 3.19. Solve the equation $\cos x = xe^x$, by $x_0 = 0.5$ by Newton-Raphson method.

Solution: $[x_2 = x_3 = 0.51776]$

Example 3.20. Solve the equation $xe^{-2x} = 0.5 \sin x$ by Newton-Raphson method.

Solution: $[x_2 = x_3 = 3.12962$ upto 5th decimal$]$

Example 3.21. Solve the equation $2x - \log_{10} x = 7$ by Newton-Raphson method.

Solution: $\hspace{8cm}$ $[x_5 = x_6 = 3.78928$ upto 5th decimal$]$

Example 3.22. Find the Newton-Raphson formula to find the value of $\dfrac{1}{N}$ where N is a real number, hence evaluate $\dfrac{1}{26}$ correct to 4 decimal places.

Solution: Let $x = \dfrac{1}{N}$

$\Rightarrow N = \dfrac{1}{x} \Rightarrow \dfrac{1}{x} - N = 0$

Let $f(x) = \dfrac{1}{x} - N \Rightarrow f'(x) = -\dfrac{1}{x^2}$

By Newton's formula ,

$$x_{n+1} = x_n - \frac{\dfrac{1}{x_n} - N}{-\dfrac{1}{x_n^2}} = x_n + \left(\dfrac{1}{x_n} - N\right)x_n^2 = x_n(2 - Nx_n)$$

This is the Newton's-Raphson formula to find the value of $\dfrac{1}{N}$.

To find $\dfrac{1}{26}$

Put $N = 26$ and $n = 0$ in $x_{n+1} = x_n(2 - Nx_n)$, we get

$$x_1 = x_0(2 - 26x_0) \hspace{6cm} (1)$$

Take $\dfrac{1}{25} = 0.04$ as x_0 $\hspace{5cm}$ $\left(\because \dfrac{1}{25} \text{ is nearer to } \dfrac{1}{26}\right)$

$$\therefore (1) \Rightarrow x_1 = x_0(2 - 26x_0) = 0.04[2 - 26(0.04)] = 0.0384$$

$$x_2 = x_1(2 - 26x_1) = 0.0384[2 - 26(0.0384)] = 0.03846$$

$$x_3 = x_2(2 - 26x_2) = 0.03846[2 - 26(0.03846)] = 0.03846$$

$\therefore$ A root of given equation is 0.03846 or 0.0385

Example 3.23. Find the Newton-Raphson formula to find the value of $\sqrt{N}$ where N is a real number, hence evaluate $\sqrt{142}$ correct to 5 decimal places.

Solution: $\hspace{8cm}$ $[x_2 = x_3 = 11.91638]$

Example 3.24. Find the Newton-Raphson formula to find the value of $N^{\frac{1}{p}}$ where N is a real number, hence evaluate $\sqrt[3]{17}$ correct to 5 decimal places.

Solution: $\hspace{8cm}$ $[x_4 = x_5 = 2.57128]$

3.4.3 Anna University Questions

1. Solve for a positive root of the equation $x^4 - x - 10 = 0$ using Newton - Raphson method.(MJ2010)

   Solution : $\hspace{7cm}$ $[x_2 = x_3 = 1.856]$

2. Find the approximate root of $xe^x = 3$ by Newton's method correcot to 3 decimal places. (MJ2011)

   Solution : $\hspace{7cm}$ $[x_2 = x_3 = 1.050]$

3. Find the Newton's iterative formula to calculate the reciprocal of N and hence find the value of $\dfrac{1}{23}$. (ND2012)

   Solution : $[x_0 = x_1 = 0.0435]$

4. Using Newton's method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.(ND2013)

   Solution : $[x_3 = x_4 = 2.740646096 \equiv 2.74065]$

5. Find by Newton-Raphson method a positive root of the equation $3x - \cos x - 1 = 0$. (ND14)

   Solution : $[x = 0.607102]$

6. Find Newton's iterative formula for the reciprocal of a number N and hence find the value of $\dfrac{1}{23}$, correct to five decimal places.

   Solution : $[x = 0.04347]$

7. Prove the quadratic convergence of Newton's-Raphson method. Find a positive root of $f(x) = x^3 - 5x + 3 = 0$, using this method.

   Solution : $[x = 0.6566]$

8. Use Newton's method to find the real root of $3x - \cos x - 1 = 0$.

   Solution : $[x = 0.607102]$

9. Obtain the positive root of $2x^3 - 3x - 6 = 0$ that lies between 1 and 2 by using Newton's-Raphson method.

   Solution : $[1.7838]$

Solution of linear system of equations

Let us consider a system of n linear algebraic equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where the coefficients a_{ij} and the constants b_i are real and known. This system of equations in matrix form may be written as

$$AX = b$$

where $A = \left(a_{ij}\right)_{n \times n}$

$\quad X = (x_1, x_2, \cdots, x_n)^T$

and $b = (b_1, b_2, \cdots, b_n)^T$.

A is called the coefficient matrix.

We are interested in finding the values $x_i, i = 1, 2, \cdots, n$ if they exist, satisfying Equation.

Methods to solve linear system of equations:

1. Gaussian elimination method

2. Pivoting-Gauss Jordan method

3.5 Gauss elimination method

Step 1: Write the augmented matrix for the given system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & b_{33} & b_3 \end{pmatrix}$$

Step 2: Using elementary row operations reduce the given matrix into an upper-triangular matrix say

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & d_1 \\ 0 & c_{22} & c_{23} & d_2 \\ 0 & 0 & c_{33} & d_3 \end{pmatrix}$$

Step 3: By back substitution we get the values for unknowns.

3.5.1 Part A

1. By Gauss elimination method solve $x + y = 2, 2x + 3y = 5$. (MJ2011)

   Solution : Given $x + y = 2$

   $$2x + 3y = 5$$

   The given system can be written as

   $$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

   $$AX = B$$

   $\therefore$ The augmented matrix is $[A|B] = \begin{bmatrix} 1 & 1 & | & 2 \\ 2 & 3 & | & 5 \end{bmatrix}$

   $$\sim \begin{bmatrix} 1 & 1 & | & 2 \\ 0 & 1 & | & 1 \end{bmatrix} R_2 \rightarrow R_2 - 2R_1 \qquad (1)$$

   Now From (1), by back substitution we get

   $$0x + y = 1 \Rightarrow \boxed{y = 1}$$

   $$x + y = 2 \Rightarrow x + 1 = 2 \Rightarrow \boxed{x = 1}$$

   $\therefore x = 1, y = 1$.

2. Solve the equations $x + 2y = 1$ and $3x - 2y = 7$ by Gauss-Elimination method. (ND2013)

Solution : $\left[x = 2, y = -\dfrac{1}{2}\right].$

3. Using Gauss elimination method solve : $5x + 4y = 15, 3x + 7y = 12.$ (AM2014)

Solution : Given $5x + 4y = 15$

$$3x + 7y = 12$$

The given system can be written as

$$\begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$$

$$AX = B$$

$\therefore$ The augmented matrix is $[A|B] = \begin{bmatrix} 5 & 4 & | & 15 \\ 3 & 7 & | & 12 \end{bmatrix}$

$$\sim \begin{bmatrix} 5 & 4 & | & 15 \\ 3 - \dfrac{3}{5}(5) & 7 - \dfrac{3}{5}(4) & | & 12 - \dfrac{3}{5}(15) \end{bmatrix} R_2 \rightarrow R_2 - \dfrac{3}{5}(R_1)$$

$$\sim \begin{bmatrix} 5 & 4 & | & 15 \\ 0 & \dfrac{23}{5} & | & \dfrac{15}{5} \end{bmatrix} \qquad (1)$$

Now From (1), by back substitution we get

$$0x + \dfrac{23}{5}y = \dfrac{15}{5} \Rightarrow \boxed{y = \dfrac{15}{23}}$$

$$5x + 4y = 15 \Rightarrow 5x = \dfrac{285}{23} \Rightarrow \boxed{x = \dfrac{57}{23}}$$

$$\therefore x = \dfrac{57}{23}, y = \dfrac{15}{23}.$$

3.5.2 Part B

Example 3.25. Solve the system of equations by Gauss elimination method

$$-3x_1 + 2x_2 - 3x_3 = -6$$

$$x_1 - x_2 + x_3 = 1$$

$$2x_1 - 5x_2 + 4x_3 = 5.$$

Solution: The given system of equations can be written as

$$x_1 - x_2 + x_3 = 1$$

$$-3x_1 + 2x_2 - 3x_3 = -6$$

$$2x_1 - 5x_2 + 4x_3 = 5$$

Given system of the form $AX = B$

Gauss elimination method:

The augmented matrix form is

$$[A|B] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ -3 & 2 & -3 & -6 \\ 2 & -5 & 4 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ -3+3 & 2-3 & -3+3 & -6+3 \\ 2-2 & -5+2 & 4-2 & 5-2 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3 & 2 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & -3+3 & 2-0 & 3+9 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - 3R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 2 & 12 \end{bmatrix}$$

$\therefore$ By back substitution, $(-2, 3, 6)$ is the solution for the system of equations.

Example 3.26. Solve the system of equations by Gauss elimination method $10x - 2y + 3z = 23$, $2x + 10y - 5z = -33$, $3x - 4y + 10z = 41$.

Solution: The matrix form is $\begin{pmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 23 \\ -33 \\ 41 \end{pmatrix}$

$$\text{i.e., } AX = B$$

The augmented matrix

$$[A, B] = \begin{pmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{pmatrix}$$

$$\sim \begin{pmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{pmatrix} \begin{matrix} \\ R_2 \to 5R_2 - R_1 \\ R_3 \to 10R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{pmatrix} \begin{matrix} \\ \\ R_3 \to 52R_3 + 34R_2 \end{matrix}$$

The reduced system of equations is given by

$$10x - 2y + 3z = 23 \tag{1}$$

$$52y - 28z = -188 \tag{2}$$

$$3780z = 11340 \tag{3}$$

From (3) $\Rightarrow z = \dfrac{11340}{3780} = 3$ $\qquad\qquad$ ($\because$ by back substitution)

From (2) $\Rightarrow 52y - 28(3) = -188$

$$52y = -104$$

$$y = -2$$

From (1) $\Rightarrow 10x + 4 + 9 = 23$

$$10x = 10$$

$$x = 1$$

Hence the solution is $x = 1, y = -2, z = 3$.

Example 3.27. Using Gauss elimination method, solve the system $3.15x - 1.96y + 3.85z = 12.95$, $2.13x + 5.12y - 2.89z = -8.61$, $5.92x + 3.05y + 2.15z = 6.88$.

Solution: The matrix form is $\begin{pmatrix} 3.15 & -1.96 & 3.85 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12.95 \\ -8.61 \\ 6.88 \end{pmatrix}$

$$\text{i.e., } AX = B$$

The augmented matrix

$$[A, B] = \begin{pmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{pmatrix}$$

$$\sim \begin{pmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 21.2107 & -16.0195 & -54.992 \end{pmatrix} \begin{matrix} \\ R_2 \to 3.15R_2 - 2.13R_1 \\ R_3 \to 3.5R_3 - 5.92R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 0 & 41.7892 & 43.8398 \end{pmatrix} \begin{matrix} \\ \\ R_3 \to 20.3028R_3 - 21.2107R_2 \end{matrix}$$

The reduced system of equations is given by

$$3.15x - 1.96y + 3.85z = 12.95 \tag{1}$$

$$20.3028y - 17.304z = -54.705 \tag{2}$$

$$41.7892z = 43.8398 \tag{3}$$

From (3) $\Rightarrow z = \dfrac{43.8398}{41.7892} = 1.049$(correct to 3 decimals)

From (2) $\Rightarrow 20.3028y - 17.304(1.049) = -54.705$

$$\Rightarrow y = -\dfrac{36.5531}{20.3028} = -1.800$$

From (1) $\Rightarrow 3.15x - 1.96(-1.8) + 3.85(1.049) = 12.95$

$$3.15x + 7.5667 = 12.95$$

$$x = \dfrac{5.3834}{3.15} = 1.709$$

Hence the solution is $x = 1.709, y = -1.8, z = 1.049$.

3.5.3 Anna University Questions

1. Solve the given system of equations by Gauss elimination method: $-x_1 + x_2 + 10x_3 = 35.61, 10x_1 + x_2 - x_3 = 11.19, x_1 + 10x_2 + x_3 = 20.08$.
   Solution:
   $$\left[x = \dfrac{1453}{1100}, y = \dfrac{837}{550}, z = \dfrac{779}{220} \right]$$

3.6 Pivoting - Gauss Jordan method

Step 1: Write the augmented matrix for the given system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & b_{33} & b_3 \end{pmatrix}$$

Step 2: Using elementary row operations reduce the given matrix into a diagonal matrix say

$$\begin{pmatrix} c_{11} & 0 & 0 & d_1 \\ 0 & c_{22} & 0 & d_2 \\ 0 & 0 & c_{33} & d_3 \end{pmatrix}$$

$$\left[\text{Here} \begin{pmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix} \text{is a} \begin{pmatrix} \text{diagonal matrix} \\ \text{or} \\ \text{unit matrix} \end{pmatrix} \right]$$

Step 2: By direct substitution we get the values for unknowns.

3.6.1 Part A

1. **For solving a linear system** $AX = B$**, compare Gauss elimination method and Gauss Jordan method.**
   Solution:

Gauss elimination method	Gauss Jordan method
Coefficient matrix is transmitted into upper triangular matrix	Coefficient matrix is transmitted into diagonal matrix or unit matrix
Direct method	Direct method
Obtain the solution by back substitution method	No need of back substitution method

2. **State the two difference between direct and iterative methods for solving system of equations.**

   Solution:

Direct Method	Iterative Method
It gives exact value	It gives only approximate value
Simple, takes less time	Time consuming and labourious
This method determine all the roots at the same time	This method determine only one root at a time

3.6.2 Part B

Example 3.28. Solve the system of equations by Gauss Jordan method

$$-3x_1 + 2x_2 - 3x_3 = -6$$
$$x_1 - x_2 + x_3 = 1$$
$$2x_1 - 5x_2 + 4x_3 = 5.$$

Solution: The given system of equations can be written as

$$x_1 - x_2 + x_3 = 1$$
$$-3x_1 + 2x_2 - 3x_3 = -6$$
$$2x_1 - 5x_2 + 4x_3 = 5$$

Given system of the form $AX = B$

Gauss-Jordan method:

The augmented matrix form is

$$[A|B] = \begin{bmatrix} 1 & -1 & 1 & | & 1 \\ -3 & 2 & -3 & | & -6 \\ 2 & -5 & 4 & | & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & | & 1 \\ -3+3 & 2-3 & -3+3 & | & -6+3 \\ 2-2 & -5+2 & 4-2 & | & 5-2 \end{bmatrix} \begin{matrix} \\ R_2 \rightarrow R_2 + 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & | & 1 \\ 0 & -1 & 0 & | & -3 \\ 0 & -3 & 2 & | & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & -3 & 2 \end{bmatrix} \begin{vmatrix} 1 \\ 3 \\ 3 \end{vmatrix} R_2 \to R_2 \times (-1)$$

$$\sim \begin{bmatrix} 1+0 & -1+1 & 1+0 \\ 0 & 1 & 0 \\ 0 & -3+3(1) & 2+3(0) \end{bmatrix} \begin{vmatrix} 1+3 \\ 3 \\ 3+3(3) \end{vmatrix} \begin{matrix} R_1 \to R_1 + R_2 \\ \\ R_3 \to R_3 + 3R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{vmatrix} 4 \\ 3 \\ 12 \end{vmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} 4 \\ 3 \\ 6 \end{vmatrix} R_3 \to \frac{R_3}{2}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{vmatrix} -2 \\ 3 \\ 6 \end{vmatrix} R_1 \to R_1 - R_3$$

$\therefore x_1 = -2, x_2 = 3, x_3 = 6$

$\Rightarrow (x_1, x_2, x_3) = (-2, 3, 6).$

Example 3.29. Solve the system of equation by (i) Gauss elimination method (ii) Gauss Jordan method

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

Solution: $[(x, y, z) = (0.99359, 1.50697, 1.84855)]$

Example 3.30. Solve the system of equation by (i) Gauss elimination method (ii) Gauss Jordan method

$$p + q + r + s = 2$$
$$p + q + 3r - 2s = -6$$
$$2p + 3q - r + 2s = 7$$
$$p + 2q + r - s = -2$$

Solution: $[(p, q, r, s) = (1, 0, -1, 2)]$

3.6.3 Anna University Questions

1. Apply Gauss-Jordan method to solve the following system of equations $x + y + z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40$. (AM11)

   Solution: $[(x, y, z) = (1, 3, 5)]$

2. Apply Gauss-Jordan method to find the solution of the following system: $10x+y+z = 12$, $2x+10y+z = 13$, $x + y + 5z = 7$ (ND11)

   Solution: $[(x, y, z) = (1, 1, 1)]$

3. Solve the system of equations by Gauss-Jordan method: $5x_1 - x_2 = 9$; $-x_1 + 5x_2 - x_3 = 4$; $-x_2 + 5x_3 = -6$

   (AM14)

   Solution: $[(x, y, z) = (2, 1, -1)]$

4. Using Gauss-Jordan method to solve $2x - y + 3z = 8$; $-x + 2y + z = 4$, $3x + y - 4z = 0$.   (ND14)

   Solution: $[x = 2, y = 2, z = 2]$

5. Solve by Gauss Jordan method, the following system $10x+y-z = 11.19$; $x+10y+z = 20.08$; $-x+y+10z = 35.61$.

   Solution: $\left[x = \dfrac{1453}{1100}, y = \dfrac{837}{550}, z = \dfrac{779}{220} \right]$

6. Apply Gauss Jordan method to solve the equations $x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$.

   Solution: $\left[x = \dfrac{11}{12}, y = \dfrac{19}{6}, z = \dfrac{59}{12} \right]$

Diagonal system

In the system of linear equations in n unknowns $AX = B$. If the coefficient matrix A is diagonally dominant then the system is said to be a diagonal system.

Thus the system of equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

i.e., $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

i.e., $AX = B$

is a diagonal system if

$$|a_1| \geq |b_1| + |c_1|$$
$$|b_2| \geq |a_2| + |c_2|$$
$$|c_3| \geq |a_3| + |b_3|$$

Note : For the Gauss Seidal method to converge quickly, the coefficient matrix A must be diagonally dominant.

If the coefficient matrix A is not diagonally dominant we must rearrange the equations in such a way that the resulting matrix becomes dominant, and then only we can apply Gauss Seidel method.

Iterative Methods:

Let us consider a system of n linear algebraic equations in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where the coefficients a_{ij} and the constants b_i are real and known. This system of equations in matrix form may be written as

$$AX = b$$

where $A = \left(a_{ij}\right)_{n \times n}$

$\qquad X = (x_1, x_2, \cdots, x_n)^T$

and $b = (b_1, b_2, \cdots, b_n)^T$.

A is called the coefficient matrix.

We are interested in finding the values $x_i, i = 1, 2n$ if they exist, satisfying Equation.

A square matrix A is called **diagonally dominant** if $|A_{ii}| \geq \sum\limits_{j \neq i} |A_{ij}|$ for all i.

i.e.,

$$|A_{11}| \geq \sum_{j \neq 1} |A_{1j}|$$

$$|A_{22}| \geq \sum_{j \neq 2} |A_{2j}|$$

$$|A_{33}| \geq \sum_{j \neq 3} |A_{3j}|$$

A square matrix A is called **strictly diagonally dominant** if $|A_{ii}| > \sum\limits_{j \neq i} |A_{ij}|$ for all i.

i.e.,

$$|A_{11}| > \sum_{j \neq 1} |A_{1j}|$$

$$|A_{22}| > \sum_{j \neq 2} |A_{2j}|$$

$$|A_{33}| > \sum_{j \neq 3} |A_{3j}|$$

The following methods to solve the system of equations by iterative methods:

(a) Gauss Jacobi Method

(a) Gauss-Seidel Method

3.7 **Iterative methods of Gauss-Jacobi**

Step 1: Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

with diagonally dominant.

Step 2: The above system should write into the form

$$x = \frac{1}{a_{11}}\left(b_1 - a_{12}y - a_{13}z\right) \tag{1}$$

$$y = \frac{1}{a_{22}}\left(b_2 - a_{21}x - a_{23}z\right) \tag{2}$$

$$z = \frac{1}{a_{33}}\left(b_3 - a_{31}x - a_{32}y\right) \tag{3}$$

Step 3: Start with the initial values $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$ for x, y, z and get $x^{(1)}, y^{(1)}, z^{(1)}$

$\therefore$ (1),(2),(3) become

$$x^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}y^{(0)} - a_{13}z^{(0)}\right)$$

$$y^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x^{(0)} - a_{23}z^{(0)}\right)$$

$$z^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x^{(0)} - a_{32}y^{(0)}\right)$$

Step 4: Using this $x^{(1)}$ for x, $y^{(1)}$ for y, $z^{(1)}$ for z in (1),(2),(3) respectively, we get

$$x^{(2)} = \frac{1}{a_{11}}\left(b_1 - a_{12}y^{(1)} - a_{13}z^{(1)}\right)$$

$$y^{(2)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x^{(1)} - a_{23}z^{(1)}\right)$$

$$z^{(2)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x^{(1)} - a_{32}y^{(1)}\right)$$

Continuing in the same procedure until the convergence is confirmed.

The general iterative formula of Gauss-Jacobi is

$$x_1^{(j+1)} = \frac{1}{a_{11}}\left[b_1 - \left(a_{12}x_2^{(j)} + a_{13}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$x_2^{(j+1)} = \frac{1}{a_{22}}\left[b_2 - \left(a_{21}x_1^{(j)} + a_{23}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$x_3^{(j+1)} = \frac{1}{a_{33}}\left[b_3 - \left(a_{31}x_1^{(j)} + a_{32}x_2^{(j)} + \cdots + a_{2n}x_n^{(j)}\right)\right]$$

$$\vdots$$

$$x_n^{(j+1)} = \frac{1}{a_{nn}}\left[b_n - \left(a_{n1}x_1^{(j)} + a_{n2}x_2^{(j)} + \cdots + a_{n,n-1}x_{n-1}^{(j)}\right)\right]$$

Note : Suppose n equations with n unknown variables $x_1, x_2, \cdots, x_n$, then

In Gauss Jacobi iteration method, for $x_n^{(j+1)}$, use $x_n^{(j)}$ values only.   [Start with $x_1 = x_2 = \cdots = 0$]

In Gauss Seidel iteration method, for $x_n^{(j+1)}$, use latest values of $x_n^{(j)}$ or $x_n^{(j+1)}$.   [Start with $x_2 = x_3 = \cdots = 0$]

3.7.1 Part A

1. Why use Jacobi ?

   Solution : Because you can separate the n−equations into n independent tasks; it is very well suited to computers with parallel processors.

3.7.2 Part B

Example 3.31. **Solve by Gauss-Jacobi iterative method**

$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$

Solution: This can be written in diagonally dominant matrix as

$$27x + 6y - z = 85 \tag{1}$$
$$6x + 15y + 2z = 72 \tag{2}$$
$$x + y + 54z = 110 \tag{3}$$
$$(1) \Rightarrow x = \frac{1}{27}(85 - 6y + z) \tag{4}$$
$$(2) \Rightarrow y = \frac{1}{15}(72 - 6x - 2z) \tag{5}$$
$$(3) \Rightarrow z = \frac{1}{54}(110 - x - y) \tag{6}$$

Iteration 1: Put the initial values $x^{(0)} = y^{(0)} = z^{(0)} = 0$ in RHS of (4),(5),(6)

$$(4) \Rightarrow x^{(1)} = \frac{1}{27}\left[85 - 6y^{(0)} + z^{(0)}\right] = \frac{1}{27}[85 - 6(0) + 0] = 3.14815$$
$$(5) \Rightarrow y^{(1)} = \frac{1}{15}\left[72 - 6x^{(0)} - 2z^{(0)}\right] = \frac{1}{15}[72 - 6(0) - 2(0)] = 4.80000$$
$$(6) \Rightarrow z^{(1)} = \frac{1}{54}\left(110 - x^{(0)} - y^{(0)}\right) = \frac{1}{54}(110 - 0 - 0) = 2.03704$$

Iteration 2: Put the intial values $x = 3.14815, y = 4.80000, z = 2.03704$ in RHS of (4),(5),(6)

$$(4) \Rightarrow x^{(2)} = \frac{1}{27}\left[85 - 6y^{(1)} + z^{(1)}\right] = \frac{1}{27}[85 - 6(4.80000) + 2.03704] = 2.15693$$
$$(5) \Rightarrow y^{(2)} = \frac{1}{15}\left[72 - 6x^{(1)} - 2z^{(1)}\right] = \frac{1}{15}[72 - 6(3.14815) - 2(2.03704)] = 3.26914$$
$$(6) \Rightarrow z^{(2)} = \frac{1}{54}\left[110 - x^{(1)} - y^{(1)}\right] = \frac{1}{54}(110 - 3.14815 - 4.80000) = 1.88985$$

Iteration 3:

$$(4) \Rightarrow x^{(3)} = \frac{1}{27}\left[85 - 6y^{(2)} + z^{(2)}\right] = \frac{1}{27}[85 - 6(3.26914) + 1.88985] = 2.49167$$
$$(5) \Rightarrow y^{(3)} = \frac{1}{15}\left[72 - 6x^{(2)} - 2z^{(2)}\right] = \frac{1}{15}[72 - 6(2.15693) - 2(1.88985)] = 3.68525$$
$$(6) \Rightarrow z^{(3)} = \frac{1}{54}\left[110 - x^{(2)} - y^{(2)}\right] = \frac{1}{54}(110 - 2.15693 - 3.26914) = 1.93655$$

Iteration 4:

$$(4) \Rightarrow x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right] = \frac{1}{27}[85 - 6(3.68525) + 1.93655] = 2.40093$$

$$(5) \Rightarrow y^{(4)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(3)}\right] = \frac{1}{15}[72 - 6(2.49167) - 2(1.93655)] = 3.54513$$

$$(6) \Rightarrow z^{(4)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = \frac{1}{54}(110 - 2.49167 - 3.68525) = 1.92265$$

Iteration 5:

$$(4) \Rightarrow x^{(5)} = \frac{1}{27}\left[85 - 6y^{(4)} + z^{(4)}\right] = \frac{1}{27}[85 - 6(3.54513) + 1.92265] = 2.43155$$

$$(5) \Rightarrow y^{(5)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(4)}\right] = \frac{1}{15}[72 - 6(2.40093) - 2(1.92265)] = 3.58328$$

$$(6) \Rightarrow z^{(5)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right] = \frac{1}{54}(110 - 2.40093 - 3.54513) = 1.92692$$

Iteration 6:

$$(4) \Rightarrow x^{(6)} = \frac{1}{27}\left[85 - 6y^{(5)} + z^{(5)}\right] = \frac{1}{27}[85 - 6(3.58328) + 1.92692] = 2.42323$$

$$(5) \Rightarrow y^{(6)} = \frac{1}{15}\left[72 - 6x^{(5)} - 2z^{(5)}\right] = \frac{1}{15}[72 - 6(2.43155) - 2(1.92692)] = 3.57046$$

$$(6) \Rightarrow z^{(6)} = \frac{1}{54}\left[110 - x^{(5)} - y^{(5)}\right] = \frac{1}{54}(110 - 2.43155 - 3.58328) = 1.92565$$

Iteration 7:

$$(4) \Rightarrow x^{(7)} = \frac{1}{27}\left[85 - 6y^{(6)} + z^{(6)}\right] = \frac{1}{27}[85 - 6(3.57046) + 1.92565] = 2.42603$$

$$(5) \Rightarrow y^{(7)} = \frac{1}{15}\left[72 - 6x^{(6)} - 2z^{(6)}\right] = \frac{1}{15}[72 - 6(2.42323) - 2(1.92565)] = 3.57395$$

$$(6) \Rightarrow z^{(7)} = \frac{1}{54}\left[110 - x^{(6)} - y^{(6)}\right] = \frac{1}{54}(110 - 2.42323 - 3.57046) = 1.92604$$

Iteration 8:

$$(4) \Rightarrow x^{(8)} = \frac{1}{27}\left[85 - 6y^{(7)} + z^{(7)}\right] = \frac{1}{27}[85 - 6(3.57395) + 1.92604] = 2.42527$$

$$(5) \Rightarrow y^{(8)} = \frac{1}{15}\left[72 - 6x^{(7)} - 2z^{(7)}\right] = \frac{1}{15}[72 - 6(2.42603) - 2(1.92604)] = 3.57278$$

$$(6) \Rightarrow z^{(8)} = \frac{1}{54}\left[110 - x^{(7)} - y^{(7)}\right] = \frac{1}{54}(110 - 2.42603 - 3.57395) = 1.92593$$

Iteration 9:

$$(4) \Rightarrow x^{(9)} = \frac{1}{27}\left[85 - 6y^{(8)} + z^{(8)}\right] = \frac{1}{27}[85 - 6(3.57278) + 1.92593] = 2.42553$$

$$(5) \Rightarrow y^{(9)} = \frac{1}{15}\left[72 - 6x^{(8)} - 2z^{(8)}\right] = \frac{1}{15}[72 - 6(2.42527) - 2(1.92593)] = 3.57310$$

$$(6) \Rightarrow z^{(9)} = \frac{1}{54}\left[110 - x^{(8)} - y^{(8)}\right] = \frac{1}{54}(110 - 2.42527 - 3.57278) = 1.92596$$

Iteration 10:

$$(4) \Rightarrow x^{(10)} = \frac{1}{27}\left[85 - 6y^{(9)} + z^{(9)}\right] = \frac{1}{27}[85 - 6(3.57310) + 1.92596] = 2.42546$$

$$(5) \Rightarrow y^{(10)} = \frac{1}{15}\left[72 - 6x^{(9)} - 2z^{(9)}\right] = \frac{1}{15}[72 - 6(2.42553) - 2(1.92596)] = 3.57299$$

$$(6) \Rightarrow z^{(10)} = \frac{1}{54}\left[110 - x^{(9)} - y^{(9)}\right] = \frac{1}{54}(110 - 2.42553 - 3.57310) = 1.92595$$

Iteration 11:

$$(4) \Rightarrow x^{(11)} = \frac{1}{27}\left[85 - 6y^{(10)} + z^{(10)}\right] = \frac{1}{27}[85 - 6(3.57299) + 1.92595] = 2.42548$$

$$(5) \Rightarrow y^{(11)} = \frac{1}{15}\left[72 - 6x^{(10)} - 2z^{(10)}\right] = \frac{1}{15}[72 - 6(2.42546) - 2(1.92595)] = 3.57302$$

$$(6) \Rightarrow z^{(11)} = \frac{1}{54}\left[110 - x^{(10)} - y^{(10)}\right] = \frac{1}{54}(110 - 2.42546 - 3.57299) = 1.92595$$

Iteration 12:

$$(4) \Rightarrow x^{(12)} = \frac{1}{27}\left[85 - 6y^{(11)} + z^{(11)}\right] = \frac{1}{27}[85 - 6(3.57302) + 1.92595] = 2.42547$$

$$(5) \Rightarrow y^{(12)} = \frac{1}{15}\left[72 - 6x^{(11)} - 2z^{(11)}\right] = \frac{1}{15}[72 - 6(2.42548) - 2(1.92595)] = 3.57301$$

$$(6) \Rightarrow z^{(12)} = \frac{1}{54}\left[110 - x^{(11)} - y^{(11)}\right] = \frac{1}{54}(110 - 2.42548 - 3.57302) = 1.92595$$

Iteration 13:

$$(4) \Rightarrow x^{(13)} = \frac{1}{27}\left[85 - 6y^{(12)} + z^{(12)}\right] = \frac{1}{27}[85 - 6(3.57301) + 1.92595] = 2.42548$$

$$(5) \Rightarrow y^{(13)} = \frac{1}{15}\left[72 - 6x^{(12)} - 2z^{(12)}\right] = \frac{1}{15}[72 - 6(2.42547) - 2(1.92595)] = 3.57302$$

$$(6) \Rightarrow z^{(13)} = \frac{1}{54}\left[110 - x^{(12)} - y^{(12)}\right] = \frac{1}{54}(110 - 2.42547 - 3.57301) = 1.92595$$

Iteration 14:

$$(4) \Rightarrow x^{(14)} = \frac{1}{27}\left[85 - 6y^{(13)} + z^{(13)}\right] = \frac{1}{27}[85 - 6(3.57302) + 1.92595] = 2.42548$$

$$(5) \Rightarrow y^{(14)} = \frac{1}{15}\left[72 - 6x^{(13)} - 2z^{(13)}\right] = \frac{1}{15}[72 - 6(2.42548) - 2(1.92595)] = 3.57302$$

$$(6) \Rightarrow z^{(14)} = \frac{1}{54}\left[110 - x^{(13)} - y^{(13)}\right] = \frac{1}{54}(110 - 2.42548 - 3.57302) = 1.92595$$

$\therefore$ Solution is $(x, y, z) = (2.42548, 3.57302, 1.92595)$

3.8 Iterative methods of Gauss-Seidel

Step 1: Let the system of equations be

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

with diagonally dominant.

Step 2: The above system should write into the form

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z) \tag{1}$$

$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z) \tag{2}$$

$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y) \tag{3}$$

Step 3: Start with the initial values $y^{(0)} = 0, z^{(0)} = 0$ for y, z and get $x^{(1)}$ from the first equation.

$\therefore$ (1) becomes $x^{(1)} = \dfrac{1}{a_{11}} \left(b_1 - a_{12}y^{(0)} - a_{13}z^{(0)} \right)$

Step 4: Using this $x^{(1)}$ in (2), we use $z^{(0)}$ for z and $x^{(1)}$ for x instead of $x^{(0)}$, we get

$\therefore$ (2) becomes $y^{(1)} = \dfrac{1}{a_{22}} \left(b_2 - a_{21}x^{(1)} - a_{23}z^{(0)} \right)$

Step 5: Substitute $x^{(1)}, y^{(1)}$ for x, y in the third equation.

$\therefore$ (3) becomes $z^{(1)} = \dfrac{1}{a_{33}} \left(b_3 - a_{31}x^{(1)} - a_{32}y^{(1)} \right)$

Step 6: To find the values of unknowns, use the latest available values on the right side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterate values, then the next iteration will be

$$x^{(r+1)} = \frac{1}{a_{11}} \left(b_1 - a_{12}y^{(r)} - a_{13}z^{(r)} \right)$$

$$y^{(r+1)} = \frac{1}{a_{22}} \left(b_2 - a_{21}x^{(r+1)} - a_{23}z^{(r)} \right)$$

$$z^{(r+1)} = \frac{1}{a_{33}} \left(b_3 - a_{31}x^{(r+1)} - a_{32}y^{(r+1)} \right)$$

Step 7: This process of continued until the convergence is confirmed.

The general iterative formula of Gauss-Seidel is

$$x_1^{(j+1)} = \frac{1}{a_{11}} \left[b_1 - \left(a_{12}x_2^{(j)} + a_{13}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)} \right) \right]$$

$$x_2^{(j+1)} = \frac{1}{a_{22}} \left[b_2 - \left(a_{21}x_1^{(j+1)} + a_{23}x_3^{(j)} + \cdots + a_{2n}x_n^{(j)} \right) \right]$$

$$x_3^{(j+1)} = \frac{1}{a_{33}} \left[b_3 - \left(a_{31}x_1^{(j+1)} + a_{32}x_2^{(j+1)} + \cdots + a_{2n}x_n^{(j)} \right) \right]$$

$$\vdots$$

$$x_n^{(j+1)} = \frac{1}{a_{nn}} \left[b_n - \left(a_{n1}x_1^{(j+1)} + a_{n2}x_2^{(j+1)} + \cdots + a_{n,n-1}x_{n-1}^{(j+1)} \right) \right]$$

Note : If either method converges, Gauss-Seidel converges faster than Jacobi.

3.8.1 Part A

1. Write the first iteration values of x, y, z when the equations $27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110$ are solved by Gauss-Seidel method.

   Solution: Since the given equations are in the diagonally dominant and find the x, y, z as follows

$$x = \frac{1}{27} \left[85 - 6y + z \right] = \frac{1}{27} \left[85 - 6(0) + (0) \right] = \frac{85}{27} = 3.14815 \qquad \text{(by putting } y = z = 0\text{)}$$

$$y = \frac{1}{15} \left[72 - 6x - 2z \right] = \frac{1}{15} \left[72 - 6(3.14815) - 2(0) \right] = 3.54074 \qquad \text{(by putting } x = 3.14815, z = 0\text{)}$$

$$z = \frac{1}{54} \left[110 - x - y \right] = \frac{1}{54} \left[110 - 3.14815 - 3.54074 \right] = 1.91317 \quad \text{(by putting } x = 3.14815, y = 3.54074\text{)}$$

$\therefore$ First iteration values of x, y, z is $(x = 3.14815, y = 3.54074, z = 1.91317)$

3.8.2 Part B

Example 3.32. Solve by Gauss-Seidel iterative method   [AM 2011]

$$x + y + 54z = 110$$

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

Solution: This can be written in diagonally dominant matrix as

$$27x + 6y - z = 85 \tag{1}$$

$$6x + 15y + 2z = 72 \tag{2}$$

$$x + y + 54z = 110 \tag{3}$$

$$(1) \Rightarrow x = \frac{1}{27}[85 - 6y + z] \tag{4}$$

$$(2) \Rightarrow y = \frac{1}{15}[72 - 6x - 2z] \tag{5}$$

$$(3) \Rightarrow z = \frac{1}{54}[110 - x - y] \tag{5}$$

Iteration 1: Put Initial values $y^{(0)} = z^{(0)} = 0$ in (4)

$$(4) \Rightarrow x^{(1)} = \frac{1}{27}\left[85 - 6y^{(0)} + z^{(0)}\right] = \frac{1}{27}[85 - 6(0) + 0] = 3.14815$$

$$(5) \Rightarrow y^{(1)} = \frac{1}{15}\left[72 - 6x^{(1)} - 2z^{(0)}\right] = \frac{1}{15}[72 - 6(3.14815) - 2(0)] = 3.54074$$

$$(6) \Rightarrow z^{(1)} = \frac{1}{54}\left[110 - x^{(1)} - y^{(1)}\right] = \frac{1}{54}[110 - 3.14815 - 3.54074] = 1.91317$$

Iteration 2: [Substitute the latest values of x, y, z in (4), (5), (6)]

$$x^{(2)} = \frac{1}{27}\left[85 - 6y^{(1)} + z^{(1)}\right] = \frac{1}{27}[85 - 6(3.54074) + 1.91317] = 2.43218$$

$$y^{(2)} = \frac{1}{15}\left[72 - 6x^{(2)} - 2z^{(1)}\right] = \frac{1}{15}[72 - 6(2.43218) - 2(1.91317)] = 3.57204$$

$$z^{(2)} = \frac{1}{54}\left[110 - x^{(2)} - y^{(2)}\right] = \frac{1}{54}[110 - 2.43218 - 3.57204] = 1.92585$$

Iteration 3:

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = \frac{1}{27}[85 - 6(3.57204) + 1.92585] = 2.42569$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(2)}\right] = \frac{1}{15}[72 - 6(2.42569) - 2(1.92585)] = 3.57294$$

$$z^{(3)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = \frac{1}{54}[110 - 2.42569 - 3.57294] = 1.92595$$

Iteration 4:

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right] = \frac{1}{27}[85 - 6(3.57294) + 1.92595] = 2.42549$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(3)}\right] = \frac{1}{15}[72 - 6(2.42549) - 2(1.92595)] = 3.57301$$

$$z^{(4)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = \frac{1}{54}[110 - 2.42549 - 3.57301] = 1.92595$$

Iteration 5:

$$x^{(5)} = \frac{1}{27}\left[85 - 6y^{(4)} + z^{(4)}\right] = \frac{1}{27}[85 - 6(3.57301) + 1.92595] = 2.42548$$

$$y^{(5)} = \frac{1}{15}\left[72 - 6x^{(5)} - 2z^{(4)}\right] = \frac{1}{15}[72 - 6(2.42548) - 2(1.92595)] = 3.57301$$

$$z^{(5)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right] = \frac{1}{54}[110 - 2.42548 - 3.57301] = 1.92595$$

∴ Solution is $(x,\ y,\ z) = (2.42548,\ 3.57301,\ 1.92595)$

Note : Iterations of the above given problem are given as follows:

Iteration	x	y	z
1	3.14815	3.54074	1.91317
2	1.91317	3.57204	1.92585
3	2.42569	3.57294	1.92595
4	2.42549	3.57301	1.92595
5	2.42548	3.57301	1.92595

Example 3.33. Solve by Gauss-Seidel iterative method

$$8x - 3y + 2z = 20$$

$$6x + 3y + 12z = 35$$

$$4x + 11y - z = 33$$

Solution: $[(x, y, z) = (3.016, 1.985, 0.911)]$

Example 3.34. Apply Gauss-Seidel method to solve the system of equations

$20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$

Solution: $[(x, y, z) = (0.999969 \approx 1, -1064 \approx -1, 102 \approx 1)]$

Example 3.35. Solve the following system by Gauss-Seidal method :

$28x + 4y - z = 32; x + 3y + 10z = 24; 2x + 17y + 4z = 35$

Solution: $[x = 0.9936, y = 1.507, z = 1.8486]$

Example 3.36. By using Gauss-Seidel method, solve the system of equations

$6x + 3y + 12z = 35; 8x - 3y + 2z = 20; 4x + 11y - z = 33.$

Solution: $[(x, y, z) = (3.016, 1.985, 0.911)]$

Example 3.37.   By using Gauss-Seidel iteration method, solve the following system of equations upto four decimals.

$10x - 2y - z - w = 3; -2x + 10y - z - w = 15; -x - y + 10z - 2w = 27; -x - y - 2z + 10w = -9.$

Solution: $[x = 3.017, y = 1.986, z = 0.91]$

Example 3.38. Using Gauss Seidel iteration method solve the following system start with

$x = 1, y = -2, z = 3: x + 3y + 52z = 173.61, x - 27y + 2z = 71.31, 41x - 2y + 3z = 65.46.$

Solution: Given

$$x + 3y + 52z = 173.61$$

$$x - 27y + 2z = 71.31$$

$$41x - 2y + 3z = 65.46$$

Since the coefficient matrix A of the given system of equations is not diagonally dominant, we rewrite the equations

$$41x - 2y + 3z = 65.46$$

$$x - 27y + 2z = 71.31$$

$$x + 3y + 52z = 173.61$$

From the above system of equations

$$x = \frac{1}{41}[65.46 + 2y - 3z]$$

$$y = -\frac{1}{27}[71.31 - x - 2z]$$

$$z = \frac{1}{52}[173.61 - x - 3y]$$

Let the initial values be $x = 1, y = -2, z = 3$

Iteration 1:	**Iteration 2:**
$x^{(1)} = \frac{1}{41}[65.46 + 2(-2) - 3(3)] = 1.2795$	$x^{(2)} = \frac{1}{41}[65.46 + 2(-2.3715) - 3(3.4509)] = 1.2284$
$y^{(1)} = -\frac{1}{27}[71.31 - 1.2795 - 2(3)] = -2.3715$	$y^{(2)} = -\frac{1}{27}[71.31 - 1.2284 - 2(3.4509)] = -2.34$
$z^{(1)} = \frac{1}{52}[173.61 - 1.2795 - 3(-2.3715)] = 3.4509$	$z^{(2)} = \frac{1}{52}[173.61 - 1.2284 - 3(-2.34)] = 3.4477$
Iteration 3:	**Iteration 4:**
$x^{(3)} = \frac{1}{41}[65.46 + 2(-2.34) - 3(3.4477)] = 1.2302$	$x^{(4)} = \frac{1}{41}[65.46 + 2(-2.3402) - 3(3.45)] = 1.23$
$y^{(3)} = -\frac{1}{27}[71.31 - 1.2302 - 2(3.4477)] = -2.3402$	$y^{(4)} = -\frac{1}{27}[71.31 - 1.23 - 2(3.45)] = -2.34$
$z^{(3)} = \frac{1}{52}[173.61 - 1.2302 - 3(-2.3402)] = 3.45$	$z^{(4)} = \frac{1}{52}[173.61 - 1.23 - 3(-2.34)] = 3.45$

From third and fourth iterations

$$x = 1.23, y = -2.34, z = 3.45$$

3.8.3 Anna University Questions

1. Use Gauss-Seidel iterative method to obtain the solution of the equations $9x - y + 2z = 9$, $x + 10y - 2z = 15$, $2x - 2y - 13z = -17$.   (MJ10)

   Solution : $[(x, y, z)^{(5)} = (x, y, z)^{(6)} = (0.917, 1.647, 1.195)]$

2. Apply Gauss-Seidel method to solve the system of equations $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$.   (ND10)

   Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (1, -1, 1)]$

3. Solve by Gauss-Seidel iterative method $x + y + 54z = 110$, $27x + 6y - z = 85$, $6x + 15y + 2z = 72$.(AM11)

   Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (2.426, 3.573, 1.926)]$

4. Solve the following system by Gauss-Seidal method :

   $28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$ (ND11)

   Solution : $[(x, y, z)^{(5)} = (x, y, z)^{(6)} = (0.9936, 1.507, 1.8486)]$

5. Apply Gauss-Seidel method to solve the system of equations

   $20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$ (MJ12,ND10)

   Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (1, -1, 1)]$

6. Solve the following system of equations using Gauss-Seidel method:$10x+2y+z = 9$, $x+10y-z = -22$, $-2x + 3y + 10z = 22$ (ND12)

   Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (1, -2, 3)]$

7. Solve by Gauss-Seidel method, the equations $27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110$.

   (AM13)

   Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (2.426, 3.573, 1.926)]$

8. Solve the equations by Gauss-Seidel method of iteration. $10x + 2y + z = 9, x + 10y - z = -22, -2x + 3y + 10z = 22$.

   (ND13)

   Solution : $[(x, y, z)^{(3)} = (x, y, z)^{(4)} = (1, -2, 3)]$

9. Using Gauss-Seidel method, solve the following system of linear equation s $4x+2y+z = 14$; $x+5y-z = 10$; $x + y + 8z = 20$.

   (AM14)

   Solution : $[(x, y, z) = (2, 2, 2)]$

10. Apply Gauss-Seidal method to solve the system of equations $20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$.

    (ND14)

    Solution : $[(x, y, z) = (1, -1, 1)]$

3.9 Matrix Inversion by Gauss Jordan method

Given Matrix $A_{n \times n}$,
Form

$$[A_{n \times n} \mid I_{n \times n}] \quad \underset{\text{Operations}}{\overset{\text{Row}}{\sim}} \quad \left[I_{n \times n} \mid A^{-1}_{n \times n} \right]$$

3.9.1 Part A

1. Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ by Gauss-Jordan method. (ND14)

Solution : Consider $[A|I] = \begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 2 & 7 & | & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 3 & | & 1 & 0 \\ 0 & 1 & | & -2 & 1 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & | & 7 & -3 \\ 0 & 1 & | & -2 & 1 \end{bmatrix} R_1 \to R_1 - 3R_2$$

$$\therefore A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

3.9.2 Part B

Example 3.39. **Using Gauss-Jordan method, find the inverse of the matrix**

$$\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

Consider $[A|I] = \begin{bmatrix} 1 & 0 & -1 & | & 1 & 0 & 0 \\ 3 & 4 & 5 & | & 0 & 1 & 0 \\ 0 & -6 & -7 & | & 0 & 0 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & -1 & | & 1 & 0 & 0 \\ 3-3(1) & 4-3(0) & 5-3(-1) & | & 0-3(1) & 1-3(0) & 0-3(0) \\ 0 & -6 & -7 & | & 0 & 0 & 1 \end{bmatrix} R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & | & 1 & 0 & 0 \\ 0 & 4 & 8 & | & -3 & 1 & 0 \\ 0 & -6 & -7 & | & 0 & 0 & 1 \end{bmatrix} R_2 \to \frac{R_2}{4}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & -3/4 & 1/4 & 0 \\ 0 & -6 & -7 & | & 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 + 6(R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & -3/4 & 1/4 & 0 \\ 0 & -6+6(1) & -7+6(2) & | & 0+6(-3/4) & 0+6(1/4) & 1+6(0) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & | & 1 & 0 & 0 \\ 0 & 1 & 2 & | & -3/4 & 1/4 & 0 \\ 0 & 0 & 5 & | & -9/2 & 3/2 & 1 \end{bmatrix} R_3 \to \frac{R_3}{5}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -3/4 & 1/4 & 0 \\ 0 & 0 & 1 & -9/10 & 3/10 & 1/5 \end{bmatrix} \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2(R_3) \end{array}$$

$$\sim \begin{bmatrix} 1+0 & 0+0 & -1+1 & 1+(-9/10) & 0+(3/10) & 0+(1/5) \\ 0-2(0) & 1-2(0) & 2-2(1) & (-3/4)-2(-9/10) & (1/4)-2(3/10) & 0-2(1/5) \\ 0 & 0 & 1 & -9/10 & 3/10 & 1/5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1/10 & 3/10 & 1/5 \\ 0 & 1 & 0 & 21/20 & -7/20 & -2/5 \\ 0 & 0 & 1 & -9/10 & 3/10 & 1/5 \end{bmatrix}$$

$$= \left[I | A^{-1} \right]$$

$$\therefore A^{-1} = \frac{1}{10} \begin{bmatrix} 1 & 3 & 2 \\ 21/2 & -7/2 & -4 \\ -9 & 3 & 2 \end{bmatrix}$$

Example 3.40. **Find the inverse of a matrix** $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ **by Gauss-Jordan method.**

Solution: $\left[A^{-1} = \frac{1}{4} \begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix} \right]$

Example 3.41. **Using Gauss-Jordan method, find the inverse of the matrix** $\begin{bmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & -8 \end{bmatrix}$

Solution: $\left[A^{-1} = \frac{1}{56} \begin{bmatrix} 12 & 4 & 6 \\ 1 & 5 & -3 \\ 5 & -3 & -1 \end{bmatrix} \right]$

Example 3.42. **Find the inverse of the matrix** $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix}$ **by using Gauss-Jordan method.**

Solution: $\left[A^{-1} = \begin{bmatrix} -1/5 & 1/3 & -1/15 \\ 4/5 & -1/3 & 4/15 \\ 2/5 & -1/3 & 7/15 \end{bmatrix} \right]$

Example 3.43. **Find the inverse of the matrix by Gauss-Jordan method:** $A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$.

Solution: $\left[A^{-1} = \begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -2 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix} \right]$

3.9.3 Anna University Questions

1. Find the inverse of the matrix by Gauss - Jordan method: $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.   (AM10)

Solution: $\left\{ A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} \right\}$

2. Find the inverse of $A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$ by using Gauss-Jordan method.   (ND10)

Solution: $\left\{ A^{-1} = \begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -3 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix} \right\}$

3. Using Gauss Jordan method, find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{pmatrix}$.   (MJ12)

Solution: $\left\{ A^{-1} = \frac{1}{4} \begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix} \right\}$

4. Using Gauss-Jordan method, find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{pmatrix}$.   (ND12)

Solution: $\left\{ A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix} \right\}$

5. Find, by Gauss-Jordan method, the inverse of the matrix $A = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{pmatrix}$.   (MJ13)

Solution: $\left\{ A^{-1} = \begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -3 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix} \right\}$

6. Find the inverse of the matrix $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ using Gauss-JOrdan method.   (ND13)

Solution: $\left\{ A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \right\}$

7. Using Gauss-Jordan method, find the inverse of $\begin{pmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & 8 \end{pmatrix}$.   (AM14)

Solution: $\left\{ A^{-1} = \begin{bmatrix} 0 & 1/5 & 3/20 \\ 1/8 & 1/40 & -3/40 \\ 1/8 & -3/40 & -1/40 \end{bmatrix} \right\}$

3.10 Eigen values of a matrix by Power method

Suppose a given square matrix is A.

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial eigen vector of the matrix A.

$$AX_1 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \left(\begin{array}{c} \text{numerically largest value} \\ \text{of matrix } A \text{ say} \\ \text{value 1} \end{array} \right) \begin{bmatrix} 1 \\ \text{value 2/value 1} \\ \text{value 3/value 1} \end{bmatrix} = \lambda_1 [X_2]$$

$$AX_2 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \left(\begin{array}{c} \text{numerically largest value} \\ \text{of matrix } A \text{ say} \\ \text{value 2} \end{array} \right) \begin{bmatrix} \text{value 1/value 2} \\ 1 \\ \text{value 3/value 2} \end{bmatrix} = \lambda_2 [X_3]$$

$\vdots$

$$\left. \begin{array}{l} AX_n = \lambda_n [X_{n+1}] \\ AX_{n+1} = \lambda_{n+1} [X_{n+2}] \end{array} \right\} \quad \begin{array}{l} \text{If } \lambda_n = \lambda_{n+1} \text{ and } X_{n+1} = X_{n+2} \text{ are same} \\ \text{upto required decimals,then stop iteration.} \end{array}$$

$\therefore$ Dominant eigen value $= \lambda_n \, (or) \, \lambda_{n+1}$ and the corresponding

eigen vector $= X_{n+1} \, (or) \, X_{n+2}$

To find the smallest eigen value of A

Form $B = A - \lambda I$ & $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary eigen vector of the matrix B.

$$BY_1 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \left(\begin{array}{c} \text{numerically largest value} \\ \text{of matrix } B \text{ say} \\ \text{value 1} \end{array} \right) \begin{bmatrix} 1 \\ \text{value 2/value 1} \\ \text{value 3/value 1} \end{bmatrix} = \lambda_1 [Y_2]$$

$$BY_2 = \begin{bmatrix} \text{value 1} \\ \text{value 2} \\ \text{value 3} \end{bmatrix} = \left(\begin{array}{c} \text{numerically largest value} \\ \text{of matrix } B \text{ say} \\ \text{value 2} \end{array} \right) \begin{bmatrix} \text{value 1/value 2} \\ 1 \\ \text{value 3/value 2} \end{bmatrix} = \lambda_2 [Y_3]$$

$\vdots$

$$\left. \begin{array}{l} BY_n = \lambda_n [Y_{n+1}] \\ BY_{n+1} = \lambda_{n+1} [Y_{n+2}] \end{array} \right\} \quad \begin{array}{l} \text{If } \lambda_n = \lambda_{n+1} \text{ and } Y_{n+1} = Y_{n+2} \text{ are same} \\ \text{upto required decimals,then stop iteration.} \end{array}$$

$\therefore$ The smallest eigen value of $A = \begin{cases} \text{Dominant eigen value of } B \\ + \\ \text{Dominant eigen value of } A \end{cases}$

To find third eigen value of A:

$\lambda_1 + \lambda_2 + \lambda_3 = $ Sum of the main diagonal elements.

3.10.1 Part A

1. What type of eigen value can by obtained by using power method.
   Solution: We can obtain dominant eigen value of the given matrix.

2. Write down all possible initial vectors of a matrix 2×2 order.
   Solution: The initial vectors are $(0, 1)^T, (1, 0)^T, (1, 1)^T$.

3. Find the dominant eigen value of $A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$ by power method upto two decimals and choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the initial eigen vector.

   Solution: Let $X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $$AX_1 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = 5X_2$$

   $$AX_2 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 4.8 \\ 3.4 \end{bmatrix} = 4.8\begin{bmatrix} 1 \\ 0.71 \end{bmatrix} = 4.8X_3$$

   $$AX_3 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.71 \end{bmatrix} = \begin{bmatrix} 4.71 \\ 3.13 \end{bmatrix} = 4.71\begin{bmatrix} 1 \\ 0.67 \end{bmatrix} = 4.71X_4$$

   $$AX_4 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.67 \end{bmatrix} = \begin{bmatrix} 4.67 \\ 3.01 \end{bmatrix} = 4.67\begin{bmatrix} 1 \\ 0.65 \end{bmatrix} = 4.67X_5$$

   $$AX_5 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.65 \end{bmatrix} = \begin{bmatrix} 4.65 \\ 2.95 \end{bmatrix} = 4.65\begin{bmatrix} 1 \\ 0.63 \end{bmatrix} = 4.65X_6$$

   $$AX_6 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.63 \end{bmatrix} = \begin{bmatrix} 4.63 \\ 2.89 \end{bmatrix} = 4.63\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = 4.63X_7$$

   $$AX_8 = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = \begin{bmatrix} 4.62 \\ 2.86 \end{bmatrix} = 4.62\begin{bmatrix} 1 \\ 0.62 \end{bmatrix} = 4.62X_8$$

$\therefore$ The eigen value $= \lambda = 4.62$ and the corresponding eigen vector $= X = \begin{bmatrix} 1 \\ 0.62 \end{bmatrix}$.

3.10.2 Part B

Example 3.44. Find the dominant eigen value and eigen vector of the matrix $\begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}$ by power method.

Solution: Let $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be an arbitrary initial eigen vector.

$$AX_1 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (-4 \times 1) & + & (-5 \times 0) \\ (1 \times 1) & + & (2 \times 0) \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} = -4\begin{bmatrix} 1 \\ -0.25 \end{bmatrix} = -4X_2, \text{ say}$$

$$AX_2 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.25 \end{bmatrix} = \begin{bmatrix} (-4 \times 1) & + & (-5 \times (-0.25)) \\ (1 \times 1) & + & (2 \times (-0.25)) \end{bmatrix} = \begin{bmatrix} -2.75 \\ 0.5 \end{bmatrix} = -2.75\begin{bmatrix} 1 \\ -0.18182 \end{bmatrix} = -2.75X_3$$

$$AX_3 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.18182 \end{bmatrix} = \begin{bmatrix} -3.0909 \\ 0.63636 \end{bmatrix} = -3.0909\begin{bmatrix} 1 \\ -0.20588 \end{bmatrix} = -3.0909X_4$$

$$AX_4 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.20588 \end{bmatrix} = \begin{bmatrix} -2.9706 \\ 0.58824 \end{bmatrix} = -2.9706\begin{bmatrix} 1 \\ -0.06666 \end{bmatrix} = -2.9706X_5$$

$$AX_5 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.06666 \end{bmatrix} = \begin{bmatrix} -3.6667 \\ 0.86668 \end{bmatrix} = -3.6667\begin{bmatrix} 1 \\ -0.23637 \end{bmatrix} = -3.6667X_6$$

$$AX_6 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.23637 \end{bmatrix} = \begin{bmatrix} -2.81815 \\ 0.52726 \end{bmatrix} = -2.81815\begin{bmatrix} 1 \\ -0.18709 \end{bmatrix} = -2.81815X_7$$

$$AX_7 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.18709 \end{bmatrix} = \begin{bmatrix} -3.06455 \\ 0.62582 \end{bmatrix} = -3.06455\begin{bmatrix} 1 \\ -0.19579 \end{bmatrix} = -3.06455X_8$$

$$AX_8 = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.19579 \end{bmatrix} = \begin{bmatrix} -3.02105 \\ 0.60842 \end{bmatrix} = -3.022105\begin{bmatrix} 1 \\ -0.20133 \end{bmatrix} = -3.022105X_9$$

Dominant Eigen value = $\lambda = -3$ and the corresponding

$$\text{Eigen vector } = X = \begin{bmatrix} 1 \\ -0.19529 \end{bmatrix} \approx \begin{bmatrix} 1 \\ -0.2 \end{bmatrix}$$

Example 3.45. Find by power method, the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$.

Solution: Dom. Eigen value = $\lambda = 11.72$ and Eigen vector = $X = \begin{bmatrix} 0.02404 \\ 0.42582 \\ 1 \end{bmatrix}$

Example 3.46. Find all the eigen values of the matrix $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by power method.

Solution: Dominant Eigen value of $A = \lambda_1 = 4$, Dominant Eigen value of $B = -5$

Smallest Eigen value of $A = \lambda_2 = $ Dom. Eig. val. of $A + $ Dom. Eig. val. of $B = 4 + (-5) = -1$

Other Eigen value of $A = \lambda_3 = $ Sum of the main diagonal elements of $A - (\lambda_1 + \lambda_2) = 6 - 3 = 3$

The required eigen values of the given matrix are $4, -1, 3$

3.10.3 Anna University Questions

1. Find the dominant eigen value and the corresponding eigen vector of the matrix $A = \begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(AM10)

Solution: $\left\{ \text{In } 8^{th} \text{ iteration, Dominant E. value} = 4, \text{corresponding E. vector } = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} \right\}$

2. Find, by power method, the largest eigenvalue and the corresponding eigenvector of a matrix $A = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ with initial vector $(1 \quad 1 \quad 1)^T$.

(ND10)

Solution: $\left\{ \text{In } 8^{th} \text{ iteration, Dom. E. value} = 11.663, \text{corres. E. vector } = (0.025 \quad 0.422 \quad 1)^T \right\}$

3. Using Jacobi's method, find all the eigenvalues and eigenvectors for the given matrix $\begin{pmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{pmatrix}$.

(AM11)

Solution: $\left\{ \text{E. values} = \{-2, 4, 6\}, \text{ E. vectors } = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$

4. Find the largest eigenvalue of $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ by using Power method.

(ND11)

Solution: $\left\{ \text{Eigen values} = \{-1, \mathbf{4}, 3\}, \text{ Eigen vectors } = \begin{bmatrix} -0.9487 \\ 0.3162 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathbf{-0.8944} \\ \mathbf{-0.4472} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} -0.2357 \\ -0.2357 \\ 0.9428 \end{bmatrix} \right\}$

5. Determine the largest eigen value and the corresponding eigen vector of the matrix $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

(MJ12)

Solution: $\left\{ \text{Eigen value} = \{0.5858, 2, \mathbf{3.4142}\}, \text{ Eigen vectors } = \begin{bmatrix} 0.5 \\ \frac{1}{\sqrt{2}} \\ 0.5 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \mathbf{-0.5} \\ \frac{1}{\sqrt{2}} \\ \mathbf{-0.5} \end{bmatrix} \right\}$

6. Find all the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix}$ using Jacobi method.

(ND12)

Solution: $\left\{ \text{Eigen values} = \{-1, 1, 5\}, \text{ Eigen vectors } = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0.5 \\ -\frac{1}{\sqrt{2}} \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ \frac{1}{\sqrt{2}} \\ 0.5 \end{bmatrix} \right\}$

7. Using Jacobi method find the all eigen values and their corresponding eigen vectors of the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. **(MJ13)**

Solution: $\left\{ \text{Eigen values} = \{-1, 5\}, \text{ Eigen vectors } = \left(-\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^T, \left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right)^T \right\}$

8. Determine the largest eigenvalue and the corresponding eigenvector of a matrix $\begin{pmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{pmatrix}$ with

$(1 \quad 1 \quad 1)^T$ as initial vector by power method. **(ND13)**

Solution: $\left\{ \text{Dominant E. value} = 11.662, \text{corresponding E. vector } = (0.0229 \quad 0.3885 \quad 0.9212)^T \right\}$

9. Find the numerically largest eigen value of $A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ and its corresponding eigen vector by

power method, taking the initial eigen vector as $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ (upto three decimal places). **(AM14)**

Solution: $\left\{ \text{Dom. E. value} = 25.1822, \text{corres. E. vector } = (-0.9967 \quad -0.0449 \quad -0.0683)^T \right\}$

10. Find the numerically largest eigenvalue if $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ and the corresponding eigen vector.

(ND14)

Solution: $\left\{ \text{Dom. E. value} = 25.1822, \text{corres. E. vector } = (-0.9967 \quad -0.0449 \quad -0.0683)^T \right\}$

11. Obtain by power method the numerically largest eigen value of the matrix $\begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -21 \end{bmatrix}$.

Solution: $\left\{ \text{E. values} = -22.2315, 20, 8.2315, \text{E. vectors } = \begin{bmatrix} 0.1000 \\ 0.2001 \\ 0.9747 \end{bmatrix}, \begin{bmatrix} 0.7068 \\ -0.5823 \\ -0.4016 \end{bmatrix}, \begin{bmatrix} -0.4399 \\ -0.8797 \\ 0.1806 \end{bmatrix} \right\}$

12. Solve by power method, to find the dominant Eigen value for the following matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution: $\left\{ \text{Eigen values} = \{-2, 3, 6\}, \text{ Eigen vectors } = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}, \begin{bmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{bmatrix} \right\}$

[Assignment Problems, P.T.O.]

3.10.4 Assignment Problems

1. Solve $e^x - 3x = 0$ by the method of fixed point iteration.   (AM12)

2. Find a positive root of the equation $cos x - 3x + 1 = 0$ by using iteration method.   (AM13)

3. Solve for a positive root of the equation $x^4 - x - 10 = 0$ using Newton - Raphson method.(MJ2010)

4. Using Newton's method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.(ND2013)

5. Solve the given system of equations by Gauss elimination method: $-x_1 + x_2 + 10x_3 = 35.61$, $10x_1 + x_2 - x_3 = 11.19$, $x_1 + 10x_2 + x_3 = 20.08$.

6. Solve the system of equations by Gauss-Jordan method: $5x_1 - x_2 = 9$; $-x_1 + 5x_2 - x_3 = 4$; $-x_2 + 5x_3 = -6$

                                     (AM14)

7. Using Gauss-Jordan method to solve $2x - y + 3z = 8$; $-x + 2y + z = 4$, $3x + y - 4z = 0$.   (ND14)

8. Using Gauss-Seidel method, solve the following system of linear equation s $4x+2y+z = 14$; $x+5y-z = 10$; $x + y + 8z = 20$.   (AM14)

9. Apply Gauss-Seidal method to solve the system of equations $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$.   (ND14)

10. Find the inverse of the matrix $\begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$ using Gauss-Jordan method.   (ND13)

11. Using Gauss-Jordan method, find the inverse of $\begin{pmatrix} 2 & 2 & 6 \\ 2 & 6 & -6 \\ 4 & -8 & 8 \end{pmatrix}$.   (AM14)

12. Find the largest eigenvalue of $\begin{pmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ by using Power method.   (ND11)

13. Find the numerically largest eigen value of $A = \begin{pmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{pmatrix}$ and its corresponding eigen vector by power method, taking the initial eigen vector as $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$ (upto three decimal places).   (AM14)

4 Interpolation, Numerical Differentiation and Numerical Integration

> Lagrange's and Newton's divided difference interpolations - Newton's forward and backward difference interpolation - Approximation of derivates using interpolation polynomials - Numerical single and double integrations using Trapezoidal and Simpson's 1/3 rules.

4.1 Introduction

The process of finding the value of a function inside the given range of discrete points are called interpolation. We have

1. Interpolation with unequal intervals

2. Interpolation with equal intervals

Methods of Equal or Unequal intervals	Methods of only Equal intervals
Lagrange's interpolation (Lagrange & Inverse Lagrange)	Newton's forward difference method
Newton's divided difference interpolation	Newton's backward difference method
Cubic Splines	

4.2 Lagrange's interpolation

Lagrangian Polynomials(Equal and unequal intervals):

Let $y = f(x)$ be a function which takes the values $y = y_0, y_1, \cdots, y_n$ corresponding to $x = x_0, x_1, \cdots, x_n$.

Lagrange's interpolation formula(x given, finding y in terms of x)

$$y = y(x) = f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}y_0$$
$$+ \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}y_1 + \ldots$$
$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_{n-1})}y_n$$

Inverse Lagrange's interpolation formula(y given, finding x in terms of y)

$$x = x(y) = f(y) = \frac{(y - y_1)(y - y_2)\ldots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\ldots(y_0 - y_n)} x_0$$
$$+ \frac{(y - y_0)(y - y_2)\ldots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\ldots(y_1 - y_n)} x_1 + \ldots$$
$$+ \frac{(y - y_0)(y - y_1)\ldots(y - y_{n-1})}{(y_2 - y_0)(y_2 - y_1)\ldots(y - y_{n-1})} x_n$$

Note: Lagrange's interpolation formula can be used for equal and unequal intervals.

4.2.1 Part A

1. State Lagrange's interpolation formula for unequal intervals.   (ND11)

   Solution: Lagrange's interpolation formula for unequal intervals is

   $$y = y(x) = f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} y_0$$
   $$+ \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} y_1 + \ldots$$
   $$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_2 - x_0)(x_2 - x_1)\ldots(x - x_{n-1})} y_n$$

2. Using Lagrange's formula, find the polynomial to the given data.   (MJ13)

   $$X: \quad 0 \quad 1 \quad 3$$
   $$Y: \quad 5 \quad 6 \quad 50$$

   Solution: Lagrange's   formula   to   find   'y'   for   three   sets   of   given   values
   $(x_0 = 0, y_0 = 5), (x_1 = 1, y_1 = 6)$ and $(x_2 = 3, y_2 = 50)$

   $$y = f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$
   $$= \frac{(x - 1)(x - 3)}{(0 - 1)(0 - 3)}(5) + \frac{(x - 0)(x - 3)}{(1 - 0)(1 - 3)}(6) + \frac{(x - 0)(x - 1)}{(3 - 0)(3 - 1)}(50)$$
   $$= \frac{5}{3}(x - 1)(x - 3) - 3x(x - 3) + \frac{25}{3}x(x - 1)$$
   $$= \frac{5}{3}(x - 1)[x - 3 + 5x] - 3\left(x^2 - 3x\right)$$
   $$= \frac{5}{3}(x - 1)[6x - 3] - 3\left(x^2 - 3x\right)$$
   $$= 5(x - 1)[2x - 1] - \left(3x^2 - 9x\right)$$
   $$= 5\left[2x^2 - 3x + 1\right] - 3x^2 + 9x$$
   $$= 7x^2 - 6x + 5$$

3. Find the second degree polynomial through the points $(0, 2)$, $(2, 1)$,$(1, 0)$ using Lagrange's formula.
   (ND14)

4. What is the assumptions we make when Lagrange's formula is used?

   Solution: Lagrange's   interpolation   formula   can   be   used   whether   the   values   of x,   the
   independent variable are equally spaced or not whether the difference of y become smaller or
   not.

5. What is the disadvantage in practice in applying Lagrange's interpolation formula?

   Solution: Though Lagrange's formula is simple and easy to remember, its application is not speedy. It requires close attention to sign and there is always a chance of committing some error due to a number of positive and negative signs in the numerator and the denominator.

6. What is 'inverse interpolation'?

   Solution: Suppose we are given a table of values of x and y. Direct interpolation is the process if finding the values of y corresponding to a value of x, not present in the table. Inverse interpolation is the process of finding the values of x corresponding to a value of y, not present in the table.

7. Construct a linear interpolating polynomial given the points (x_0, y_0) and (x_1, y_1).

   Solution: $y = y(x) = f(x) = \dfrac{(x - x_1)}{(x_0 - x_1)} y_0 + \dfrac{(x - x_0)}{(x_1 - x_0)} y_1$

8. What is the Lagrange's formula to find 'y' if three sets of values $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2) are given.

   Solution: $y = \dfrac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \dfrac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \dfrac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$

9. Find the second degree polynomial fitting the following data:

x	1	2	4
y	4	5	13

Solution: Here $x_0 = 1, x_1 = 2, x_2 = 4$

$$y_0 = 4, y_2 = 5, y_2 = 13$$

By Lagrange's formula for three points is

$$y = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$

$$y = \frac{(x^2 - 6x + 8)}{(-1)(-3)} (4) + \frac{(x^2 - 5x + 4)}{(1)(-2)} (5) + \frac{(x^2 - 3x + 2)}{(3)(2)} (13)$$

$$= \frac{(x^2 - 6x + 8)}{3} (4) + \frac{(x^2 - 5x + 4)}{-2} (5) + \frac{(x^2 - 3x + 2)}{6} (13)$$

$$= \frac{1}{6} \left[8x^2 - 48x + 64 - 15x^2 + 75x - 60 + 13x^2 - 39x + 26 \right]$$

$$= \frac{1}{6} \left[6x^2 - 12x + 30 \right]$$

$$y = f(x) = x^2 - 2x + 5$$

4.2.2 Part B

Example 4.1. **Using Lagrange interpolation formula, find $f(4)$ given that $f(0) = 2, f(1) = 3, f(2) = 12, f(15) = 3587$.**

Solution: Given

x	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 15$
$y = f(x)$	$y_0 = 2$	$y_1 = 3$	$y_2 = 12$	$y_3 = 3587$

Lagrange interpolation formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$f(4) = \frac{(4-1)(4-2)(4-15)}{(0-1)(0-2)(0-15)}(2) + \frac{(4-0)(4-2)(4-15)}{(1-0)(1-2)(1-15)}(3)$$
$$+ \frac{(4-0)(4-1)(4-15)}{(2-0)(2-1)(2-15)}(12) + \frac{(4-0)(4-1)(4-2)}{(15-0)(15-1)(15-2)}(3587)$$

$$= \frac{(3)(2)(-11)}{(-1)(-2)(-15)}(2) + \frac{(4)(2)(-11)}{(1)(-1)(-14)}(3)$$
$$+ \frac{(4)(3)(-11)}{(2)(1)(-13)}(12) + \frac{(4)(3)(2)}{(15)(14)(13)}(3587)$$

$$= 77.99 = 78$$

Example 4.2. Find polynomial $f(x)$ by using Lagrange formula from the given data and find $f(8)$.

x	3	7	9	10
$f(x)$	168	120	72	63

Solution: Lagrange polynomial $f(x) = x^3 - 21x^2 + 119x - 27$.

$$[y_{(x=8)} \text{ or } y(x=8) \text{ or } f(x=8) = 93]$$

Example 4.3. Use Lagrange's formula to fit a polynomial to the data

x	−1	0	2	3
$f(x)$	−8	3	1	12

and hence find $y(1)$.

Solution: Lagrange polynomial $f(x) = 2x^3 - 6x^2 + 3x + 3$.

$$[y_{(x=1)} = y(x=1) = f(x=1) = 2.]$$

Example 4.4. Using Lagrange's formula, prove that

$$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5}).$$

Solution: From the equation, the values of x are

$x_0 = -5$	$x_1 = -3$	$x_2 = 3$	$x_3 = 5$
$y_0 = y_{-5}$	$y_1 = y_{-3}$	$y_2 = y_3$	$y_3 = y_5$

The x values are not equally space, so use Lagrange's formula to find $y = f(x)$.

Lagrange's formula for a set of 4 pair of values is

$$y = y_x = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} y_3$$

$$= \frac{(x + 3)(x - 3)(x - 5)}{(-5 + 3)(-5 - 3)(-5 - 5)} y_{-5} + \frac{(x + 5)(x - 3)(x - 5)}{(-3 + 5)(-3 - 3)(-3 - 5)} y_{-3}$$

$$+ \frac{(x + 5)(x + 3)(x - 5)}{(3 + 5)(3 + 3)(3 - 5)} y_3 + \frac{(x + 5)(x + 3)(x - 3)}{(5 + 5)(5 + 3)(5 - 3)} y_5$$

Put $x = 1$, we get

$$y_1 = \frac{(1 + 3)(1 - 3)(1 - 5)}{(-5 + 3)(-5 - 3)(-5 - 5)} y_{-5} + \frac{(1 + 5)(1 - 3)(1 - 5)}{(-3 + 5)(-3 - 3)(-3 - 5)} y_{-3}$$

$$+ \frac{(1 + 5)(1 + 3)(1 - 5)}{(3 + 5)(3 + 3)(3 - 5)} y_3 + \frac{(1 + 5)(1 + 3)(1 - 3)}{(5 + 5)(5 + 3)(5 - 3)} y_5$$

$$= -0.2 y_{-5} + 0.5 y_{-3} + y_3 - 0.3 y_5$$

$$= -0.2 y_{-5} + 0.2 y_{-3} + 0.3 y_{-3} + y_3 - 0.3 y_5$$

$$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

Example 4.5. Find the age corresponding to the annuity value 13.6 from the given table

Age (x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

Solution: $\left[x_{13.6} \text{ or } (y = 13.6) = 43 \right]$

Example 4.6. Find x for which $y = 7$, given

x	1	3	4
y	4	12	19

Solution: $\left[x(y = 7) = 1.8565 \right]$

4.2.3 Anna University Questions

1. Use Lagrange's formula to find a polynomial which takes the values $f(0) = -12$, $f(1) = 0$, $f(3) = 6$ and $f(4) = 12$. Hence find $f(2)$. (AM10)

   Solution: $\left[f(x) = x^3 - 7x^2 + 18x - 12 \Rightarrow f(2) = 4 \right]$

2. Using Lagrange's interpolation formula to fit a polynomial to the given data $f(-1) = -8$, $f(0) = 3$, $f(2) = 1$ and $f(3) = 12$. Hence find the valur of $f(1)$. (ND10)

   Solution: $\left[f(x) = 2x^3 - 6x^2 + 3x + 3 \Rightarrow f(1) = 2 \right]$

3. Find the expression of $f(x)$ using Lagrange's formula for the followin data. (AM11)

$$x: \quad 0 \quad 1 \quad 4 \quad 5$$
$$f(x): \quad 4 \quad 3 \quad 24 \quad 39$$

   Solution: $\left[f(x) = 2x^2 - 3x + 4 \right]$

4. Find the value of x when $y = 20$ using Lagrange's formula from the following table.   (AM11)

$$x : \quad 1 \quad 2 \quad 3 \quad 4$$
$$y = f(x) : \quad 1 \quad 8 \quad 27 \quad 64$$

Solution: $[f(y = 20) = x(y = 20) = 2.84675]$

5. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data:

                                                                (MJ12)

Year:	1997	1999	2001	2002
Profit in Lakhs Rs. :	43	65	159	248

Solution: $[f(x = 2000) = 100]$

6. Use Lagrange's method to find $\log_{10} 656$, given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, and $\log_{10} 661 = 2.8202$.   (ND12)

Solution: $[f(656) = 2.8168]$

7. Apply Lagrange's formula, to find $y(27)$ to the data given below.   (MJ13)

$$x : \quad 14 \quad 17 \quad 31 \quad 35$$
$$y : \quad 68.8 \quad 64 \quad 44 \quad 39.1$$

Solution: $[f(27) = 49.3]$

8. Use Lagrange's formula to find the value of y at $x = 6$ from the following data: (ND13)

$$x : \quad 3 \quad 7 \quad 9 \quad 10$$
$$y : \quad 168 \quad 120 \quad 72 \quad 63$$

Solution: $[f(6) = 147]$

9. Using Lagrange's interpolation formula, find $y(2)$ from the following data: $y(0) = 0; y(1) = 1; y(3) = 81; y(4) = 256; y(5) = 625$ (AM14)

Solution: $[f(2) = \qquad]$

10. Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for the following values of x and y: (ND14)

$$x \quad 0 \quad 1 \quad 2 \quad 5$$
$$y \quad 2 \quad 3 \quad 12 \quad 147$$

Solution: $[f(x) = \qquad , f(2) = \qquad]$

11. Find the Lagrange's polynomial of degree 3 to fit the data : $y(0) = -12$, $y(1) = 0$, $y(3) = 6$ and $y(4) = 12$. Hence find $y(2)$. $\left[\text{Ans: } f(x) = x^3 - 7x^2 + 18x - 12; y(2) = 4\right]$

12. Find the missing term in the following table using Lagrange's interpolation.

x	0	1	2	3	4
y	1	3	9	–	81

[Ans : 31]

13. Find the value of x corresponding to $y = 100$ from the table.

x	3	5	7	9	11
y	6	24	58	108	174

[Ans : 8.656]

4.3 Newton's divided difference interpolation

First divided difference for arguments x_0, x_1 :

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1) = \underset{x_1}{\triangle} f(x_0) = [x_0, x_1](or)[x_1, x_0] = \underset{x_0}{\triangle} f(x_1)$$

First divided difference for arguments x_1, x_2 :

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f(x_1, x_2) = \underset{x_2}{\triangle} f(x_1) = [x_1, x_2](or)[x_2, x_1] = \underset{x_1}{\triangle} f(x_2)$$

Second divided difference for arguments x_1, x_2, x_3 :

$$\frac{\underset{x_3}{\triangle} f(x_2) - \underset{x_2}{\triangle} f(x_1)}{x_3 - x_1} = f(x_1, x_2, x_3) = \underset{x_3 x_2}{\triangle^2} f(x_1) = [x_1, x_2, x_3]$$

Third divided difference for arguments x_0, x_1, x_2, x_3 :

$$\frac{\underset{x_3 x_2}{\triangle^2} f(x_1) - \underset{x_2 x_1}{\triangle^2} f(x_0)}{x_3 - x_0} = f(x_0, x_1, x_2, x_3) = \underset{x_3 x_2 x_1}{\triangle^3} f(x_0) = [x_0, x_1, x_2, x_3]$$

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \cdots$$

4.3.1 Properties of divided differences

1. The divided difference are symmetric functions of their arguments.

For example, $(1) \underset{y}{\triangle} f(x) = \underset{x}{\triangle} f(y)$

$(2) \underset{yz}{\triangle^2} f(x) = \underset{xz}{\triangle^2} f(y) = \underset{xy}{\triangle^2} f(z)$

2. The nth divided differences of a polynomial of degree n are constants.

3. The divided difference operator($\triangle$) is a linear operator.

$\triangle[f(x) + g(x)] = \triangle[f(x)] + \triangle[g(x)]$ and

$\triangle[cf(x)] = c\triangle[f(x)],$ where c is constant

Note : For n set of (x, y) values, we get upto $(n - 1)^{\text{th}}$ divided difference values.

4.3.2 Part A

1. Form the divided difference table for the data $(0,1)$, $(1,4)$, $(3,40)$ and $(4,85)$.   (AM10)

   Solution: Newton's divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		$\dfrac{4-1}{1-0}=3$		
1	4		$\dfrac{18-3}{3-0}=5$	
		$\dfrac{40-4}{3-1}=18$		$\dfrac{6.75-5}{4-0}=0.44$
3	40		$\dfrac{45-18}{4-0}=6.75$	
		$\dfrac{85-40}{4-3}=45$		
4	85			

2. Find the first and second divided differences with arguments a, b, c of the function $f(x) = \dfrac{1}{x}$. (ND10)

   Solution : If $f(x) = \dfrac{1}{x} \Rightarrow f(a) = \dfrac{1}{a}$

   $$f(a,b) = \Delta_b \frac{1}{a} = \frac{f(b)-f(a)}{b-a} = \frac{\dfrac{1}{b}-\dfrac{1}{a}}{b-a} = -\frac{1}{ab}$$

   $$f(a,b,c) = \Delta_{bc}^2 \frac{1}{a} = \frac{f(b,c)-f(a,b)}{c-a} = \frac{-\dfrac{1}{bc}+\dfrac{1}{ab}}{c-a} = \frac{-a+c}{abc(c-a)} = \frac{1}{abc}$$

   $$\therefore \Delta_{bc}^2 \frac{1}{a} = \frac{1}{abc}$$

3. Find the divided differences of $f(x) = x^3 - x + 2$ for the arguments $1, 3, 6, 11$.   (AM11)

   Solution: Newton's divided difference table is

x	$f(x) = x^3 - x + 2$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	2			
		$\dfrac{26-2}{3-1}=12$		
3	26		$\dfrac{62-12}{6-1}=10$	
		$\dfrac{212-26}{6-3}=62$		$\dfrac{20-10}{11-1}=1$
6	212		$\dfrac{222-62}{11-3}=20$	
		$\dfrac{1322-212}{11-6}=222$		
11	1322			

4. Construct the divided difference table for the following data: (MJ12)

   $$x: \quad 0 \quad 1 \quad 2 \quad 5$$
   $$f(x): \quad 2 \quad 3 \quad 12 \quad 147$$

   Solution: [Ref : Part A : Example 1 (AM10)]

5. Find the divided differences of $f(x) = x^3 - x^2 + 3x + 8$ for arguments $0, 1, 4, 5$. (ND13)

   Solution: [Ref : Part A : Example 3 (AM11)]

6. Find the second divided difference with arguments a, b, c, if $f(x) = \dfrac{1}{x}$. (AM14, ND10)

   Solution: [Ref : Part A : Example 2 (ND10)]

7. Prove that $\underset{yz}{\triangle^2} x^3 = x + y + z$.

   Solution : Given the function $f(x) = x^3$ and the arguments are x, y, z.

   $$\underset{y}{\triangle} f(x) = \frac{f(y) - f(x)}{y - x} = \frac{y^3 - x^3}{y - x} = \frac{(y - x)(x^2 + xy + y^2)}{y - x} = x^2 + xy + y^2$$

   Similarly, $\underset{z}{\triangle} f(y) = y^2 + yz + z^2$

   Now, $\underset{yz}{\triangle^2} f(x) = \dfrac{\underset{z}{\triangle} f(y) - \underset{y}{\triangle} f(x)}{z - x} = \dfrac{y^2 + yz + z^2 - (x^2 + xy + y^2)}{z - x}$

   $$= \frac{z^2 - x^2 + yz - xy}{z - x} = \frac{(z + x)(z - x) + y(z - x)}{z - x} = x + y + z$$

8. Show that $\underset{bcd}{\triangle^3}\left(\dfrac{1}{a}\right) = -\dfrac{1}{abcd}$

   Solution : If $f(x) = \dfrac{1}{x}$, $f(a) = \dfrac{1}{a}$

   $$f(a, b) = \underset{b}{\triangle}\frac{1}{a} = \frac{f(b) - f(a)}{b - a} = \frac{\dfrac{1}{b} - \dfrac{1}{a}}{b - a} = -\frac{1}{ab}$$

   Similarly, $f(b, c) = \underset{c}{\triangle}\dfrac{1}{b} = -\dfrac{1}{bc}$, $f(c, d) = \underset{d}{\triangle}\dfrac{1}{c} = -\dfrac{1}{cd}$

   $$f(a, b, c) = \underset{bc}{\triangle^2}\frac{1}{a} = \frac{\underset{c}{\triangle}\dfrac{1}{b} - \underset{b}{\triangle}\dfrac{1}{b}}{c - a} = \frac{-\dfrac{1}{bc} + \dfrac{1}{ab}}{c - a} = \frac{-a + c}{abc(c - a)} = \frac{1}{abc}$$

   Similarly, $f(b, c, d) = \underset{cd}{\triangle^2}\dfrac{1}{b} = \dfrac{1}{bcd}$

   $$\therefore f(a, b, c, d) = \underset{bcd}{\triangle^3}\frac{1}{a} = \frac{\underset{cd}{\triangle^2}\dfrac{1}{b} - \underset{bc}{\triangle^2}\dfrac{1}{a}}{d - a} = \frac{\dfrac{1}{bcd} - \dfrac{1}{abc}}{d - a}$$

   $$= \frac{a - d}{abc(d - a)} = -\frac{1}{abcd}$$

4.3.3 Part B

Example 4.7. **Construct the divided difference table for the following data and find the value $f(2)$.**

x	4	5	7	10	11	12
$y = f(x)$	50	102	296	800	1010	1224

Solution: Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3) + \cdots$$

Newton's divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	50					
		$\frac{102-50}{5-4} = 52$				
5	102		$\frac{97-52}{7-4} = 15$			
		$\frac{296-102}{7-5} = 97$		$\frac{14.2-15}{10-4} = -0.133$		
7	296		$\frac{168-97}{10-5} = 14.2$		$\frac{-.617+.133}{11-4} = -.069$	
		$\frac{800-296}{10-7} = 168$		$\frac{10.5-14.2}{11-5} = -.617$		$\frac{-.158+.069}{12-4} = -.011$
10	800		$\frac{210-168}{11-7} = 10.5$		$\frac{-1.7+.617}{12-5} = -.158$	
		$\frac{1010-800}{11-10} = 210$		$\frac{2-10.5}{12-7} = -1.7$		
11	1010		$\frac{214-210}{12-10} = 2$			
		$\frac{1224-1010}{12-11} = 214$				
12	1224					

$$f(x) = 50 + (x - 4)(52) + (x - 4)(x - 5)(15) + (x - 4)(x - 5)(x - 7)(-0.133)$$

$$+ (x - 4)(x - 5)(x - 7)(x - 10)(-0.069)$$

$$+ (x - 4)(x - 5)(x - 7)(x - 10)(x - 11)(-0.011)$$

$$f(x = 2) = 50 + (2 - 4)(52) + (2 - 4)(2 - 5)(15) + (2 - 4)(2 - 5)(2 - 7)(-0.133)$$

$$+ (2 - 4)(2 - 5)(2 - 7)(2 - 10)(-0.069)$$

$$+ (2 - 4)(2 - 5)(2 - 7)(2 - 10)(2 - 11)(-0.011)$$

$$= 47.19$$

Example 4.8. If $f(0) = 0, f(1) = 0, f(2) = -12, f(4) = 0, f(5) = 600, f(7) = 7308$, **find a polynomial that satisfies this data using Newton's devided difference formula. Hence find** $f(6), f(-1)$.

Solution: Hint : $f(x) = x(x - 1)[x^3 - 2x^2 - 13x + 32]$

$$f(6) = 2940$$

$$f(-1) = 84$$

Example 4.9. Find the third divided difference with arguments $2, 4, 9, 10$ **of the function** $f(x) = x^3 - 2x$.

Solution: [Ref : Part A : Example 3, 5]

4.3.4 Anna University Questions

1. Find the function $f(x)$ from the following table using Newton's divided difference formula:

        $x :$ 0   1   2   4   5   7   (AM10)

        $f(x) :$ 0   0   –12   0   600   7308

Solution: $\left[f(x) = x^5 - 3x^4 - 11x^3 + 33x^2 - 20x \Rightarrow f(6) = 2580\right]$

2. Given tha table

$$x: \quad 5 \quad 7 \quad 11 \quad 13 \quad 17$$
$$f(x): \quad 150 \quad 392 \quad 1452 \quad 2366 \quad 5202$$

Evaluate $f(9)$ using Newton's divided difference formula. (AM11)

Solution: $\left[f(x) = x^3 - x^2 + 24x - 70 \Rightarrow f(9) = 794\right]$

3. Determine $f(x)$ as a polynomial in x for the following data, using Newton's divided difference formulae. Also find $f(2)$. (ND11)

$$x: \quad -4 \quad -1 \quad 0 \quad 2 \quad 5$$
$$f(x): \quad 1245 \quad 33 \quad 5 \quad 9 \quad 1335$$

Solution: $\left[f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5 \Rightarrow f(2) = 9\right]$

4. Use Newton's divided difference formula to find $f(x)$ from the following data. (MJ13)

$$x: \quad 1 \quad 2 \quad 7 \quad 8$$
$$y: \quad 1 \quad 5 \quad 5 \quad 4$$

Solution: $\left[f(x) = \dfrac{1}{42}\left(3x^3 - 58x^2 + 321x - 224\right)\right]$

5. Find $f(3)$ by Newton's divided difference formula for the following data: (AM14, ND2004)

$$x: \quad -4 \quad -1 \quad 1 \quad 2 \quad 5$$
$$y: \quad 1245 \quad 33 \quad 5 \quad 9 \quad 1335$$

Solution: $\left[f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5 \Rightarrow f(3) = 125\right]$

6. By using Newton's divided difference formula find $f(8)$, gives (ND14)

$$x \quad 4 \quad 5 \quad 7 \quad 10 \quad 11 \quad 13$$
$$f(x) \quad 48 \quad 100 \quad 294 \quad 900 \quad 1210 \quad 2028$$

7. If $f(0) = f(1) = 0, f(2) = -12, f(4) = 0, f(5) = 600$ and $f(7) = 7308$, find a polynomial that satisfies this data using Newton's divided difference interpolation formula. Hence, find $f(6)$. (MJ2007)

$$[\text{Ans} : f(6) = 2580]$$

8. Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Evaluate $f(9)$ using Newton's divided difference formula. (N/D2007)

$$\left[\text{Ans} : f(x) = x^3 - x^2 + 24x - 70; f(9) = 794\right]$$

9. Using Newton's divided difference interpolation, find the polynomial of the given data

x	−1	0	1	3
$f(x)$	2	1	0	−1

(AU Nov/Dec 2007)

$$\left[\text{Ans} : f(x) = \dfrac{1}{24}\left(x^3 - 25x + 24\right)\right]$$

4.4 Cubic Splines

Interpolating with a cubic spline

The cubic spline interpolation formula is

$$S(x) = y(x) = y = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i]$$

$$+ \frac{1}{h}(x_i - x)[y_{i-1} - \frac{h^2}{6} M_{i-1}] + \frac{1}{h}(x - x_{i-1})[y_i - \frac{h^2}{6} M_i]$$

where

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

$$n = \text{number of data}$$

$$i = \text{number of intervals [i.e., } i = 1, 2, 3, , (n-1)]$$

$$h = \text{length of interval} = \text{ interval length.}$$

Note : If M_i and y_i'' values are not given, then assume $M_0 = M_n = 0 \left[\text{or } y_0'' = y_n'' = 0\right]$, and find $M_1, M_2, \cdots, M_{n-1}$ in 1st interval, 2nd interval, $\cdots$, $(n-1)^{\text{th}}$ interval value.

Note : Order of convergence of the cubic spline is 4.

4.4.1 Part A

1. Define a cubic spline $S(x)$ which is commonly used for interpolation.   (AM10)

   Solution: Definition(Cubic Spline Interpolation): Given a function f defined on $[a, b]$ and $a = x_0 < x_1 < \cdots < x_n = b$, a cubic spline interpolant S for f is a function that satisfies the following conditions:

   1. For each $j = 1, \cdots, n, S(x)$ is a cubic polynomial, denoted by $S_j(x)$, on the subinterval $[x_{j-1}, x_j]$.

   2. $S(x_j) = f(x_j)$ for each $j = 0, 1, \cdots, n$.

   3. $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \cdots, n - 2$.

   4. $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \cdots, n - 2$.

   5. $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \cdots, n - 2$.

   6. One of the following sets of boundary conditions is satisfied:

      i. $S''(x_0) = S''(x_n) = 0$ (natural or free boundary);

      ii. $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (clamped boundary).

2. Define cubic spline function.   (ND11)

   Solution : A function $s \in C^2[a, b]$ is called a cubic spline on $[a, b]$, if s is a cubic polynomial s_i in each interval $[x_i, x_{i+1}]$. It is called a cubic interpolating spline if $s(x_i) = y_i$ for given values y_i.

3. For cubic splines, what are the 4n conditions required to evaluate the unknowns.   (MJ12)

   Solution: We need $4n$ conditions to fix the coefficients:

   1. $S(x_j) = f(x_j)$ for each $j = 1, \cdots, n \Rightarrow$ (n conditions)

2. $S_{n?1} = y_n,$ 　　　　　　　　　　　　　　　　1 condition $\Rightarrow$ (1 condition)

3. $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$ 　　　　　　　for each $j = 1, \cdots, n-1. \Rightarrow$ ($n-1$ conditions)

4. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$ 　　　　　　for each $j = 1, \cdots, n-1. \Rightarrow$ ($n-1$ conditions)

5. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$ 　　　　　for each $j = 1, \cdots, n-1. \Rightarrow$ ($n-1$ conditions)

$\Rightarrow$ Totally $(4n - 2$ conditions)

These are $4n - 2$ conditions. We need two extra.

We can define two extra boundary conditions.

Natural Spline: 　　　　　　　$M_0 = S''_0(x_0) = 0$ and $M_n = S''_n(x_n) = 0.$ 　　　　　(2 conditions)

$\therefore 4n - 2 + 2 = 4n$ conditions required to evaluate the unknowns for cubic spline.

4. Define cubic spline. 　　　　　　　　　　　　　　　　　　　　　　　　　　(ND12, ND11)

5. What is a cubic spline?

Solution : A cubic spline which has continuous slope and curvature is called a cubic spline.

6. What is a natural cubic spline?

Solution : A cubic spline fitted to the given data such that the end cubics approach linearity at their extremities is called a natural cubic spline.

7. State the conditions required for a natural cubic spline.

Solution : A cubic spline $g(x)$ fits to each of the points is continuous and is continuous in slope and curvature such that $M_0 = S_0 = g''_0(x_0) = 0$ and $M_n = S_n = g''_{n-1}(x_n) = 0$ is called a natural cubic spline. Let us assume that $(x_i, y_i), i = 0, 1, 2, \cdots, n$ are data points.

8. What are the advantages of cubic spline fitting?

Solution : Cubic spline provide better approximation to the behavior of functions that have abrupt local changes. Further, spline perform better than higher order polynomial approximation.

9. Write the end conditions on $M_i(x)$ in natural cubic spline.

Solution : $M_0(x) = 0, M_n(x) = 0.$

10. Write the relation between the second derivatives $M_i(x)$ in cubic splines with equal mesh spacing.

Solution :

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}], i = 1, 2, \cdots, n-1.$$

Or

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[f_{i-1} - 2f_i + f_{i+1}], i = 1, 2, \cdots, n-1$$

4.4.2 Part B

Example 4.10. Find the cubic spline approximation for the function f(x) given by the data:

x	0	1	2	3
$y = f(x)$	1	2	33	244

with $M_0 = 0 = M_3$. Hence estimate the value $f(0.5), f(1.5), f(2.5)$. {AU2010}

Solution: We know that cubic spline interpolation formula for $x_{i-1} \leq x < x_i, i = 1, 2, 3$ is

$$S_i(x) = y(x) = y = \frac{1}{6h} \left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i \right]$$

$$+ \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6} M_{i-1} \right]$$

$$+ \frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6} M_i \right] \tag{1}$$

$$\text{where } M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}] \tag{2}$$

$$n = \text{number of data} = 4$$

$$i = \text{number of intervals} = 3 \text{ i.e., } i = 1, 2, 3.$$

$$h = \text{length of inteval} = 1$$

Given $M_0 = M_2 = 0$, so find M_1, M_2.

Suppose M_i or y_i'' values are not given, then assume $M_0 = M_3 = 0$ and find M_1, M_2.

<u>To find M_1, M_2</u>

$$\text{When } i = 1, (2) \Rightarrow M_0 + 4M_1 + M_2 = \frac{6}{1^2}[y_0 - 2y_1 + y_2]$$

$$\Rightarrow 0 + 4M_1 + M_2 = 6[1 - 2(2) + 33]$$

$$\Rightarrow 4M_1 + M_2 = 180 \tag{3}$$

$$\text{When } i = 2, (2) \Rightarrow M_1 + 4M_2 + M_3 = \frac{6}{1^2}[y_1 - 2y_2 + y_3]$$

$$\Rightarrow M_1 + 4M_2 + 0 = 6[2 - 2(2) + 244]$$

$$\Rightarrow 4M_1 + M_2 = 180 \tag{4}$$

$$\text{Solving (3)\&(4),} \qquad (3) \Rightarrow 4M_1 + M_2 = 180$$

$$4 \times (4) \Rightarrow 4M_1 + 16M_2 = 4320$$

$$\text{i.e., } (3) + 4 \times (4) \Rightarrow -15M_2 = 4140$$

$$\Rightarrow M_2 = 276$$

$$(3) \Rightarrow 4M_1 = 180 - 276$$

$$\Rightarrow M_1 = -24$$

<u>To find Cubic spline</u>

When $i = 1$,

Cubic spline in $x_{i-1} \leq x \leq x_i$

$$\text{i.e., } x_0 \leq x \leq x_1$$

$$\text{i.e., } 0 \leq x \leq 1$$

i.e., Cubic spline in $0 \leq x \leq 1$ is

$$y_1(x) = S_1(x) = \frac{1}{6(1)}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\right]$$

$$+ \frac{1}{1}(x_1 - x)\left[y_0 - \frac{1}{6}M_0\right] + \frac{1}{1}(x - x_0)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[(1 - x)^3(0) + (x - 0)^3(-24)\right]$$

$$+ (1 - x)[1 - 0] + (x - 0)[2 - (-24)]$$

$$= -4x^3 + (1 - x) + 6x$$

$$= -4x^3 + 5x + 1$$

When $i = 2$,

Cubic spline in $x_{i-1} \leq x \leq x_i$

i.e., $x_1 \leq x \leq x_2$

i.e., $1 \leq x \leq 2$

i.e., Cubic spline in $1 \leq x \leq 2$ is

$$y_2(x) = S_2(x) = \frac{1}{6(1)}\left[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2\right]$$

$$+ \frac{1}{1}(x_2 - x)\left[y_1 - \frac{1}{6}M_1\right] + \frac{1}{1}(x - x_1)\left[y_2 - \frac{1}{6}M_2\right]$$

$$= \frac{1}{6}\left[(2 - x)^3(-24) + (x - 1)^3(276)\right]$$

$$+ (2 - x)\left[2 - \frac{1}{6}(-24)\right] + (x - 1)\left[33 - \frac{1}{6}(276)\right]$$

$$= -4(2 - x)^3 + 46(x - 1)^3 + 6(2 - x) - 13(x - 1)$$

$$= 50x^3 - 162x^2 + 162x - 53$$

When $i = 3$,

Cubic spline in $x_{i-1} \leq x \leq x_i$

i.e., $x_2 \leq x \leq x_3$

i.e., $2 \leq x \leq 3$

i.e., Cubic spline in $2 \leq x \leq 3$ is

$$y_3(x) = S_3(x) = \frac{1}{6(1)}\left[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3\right]$$

$$+ \frac{1}{1}(x_3 - x)\left[y_2 - \frac{1}{6}M_2\right] + \frac{1}{1}(x - x_2)\left[y_3 - \frac{1}{6}M_3\right]$$

$$= \frac{1}{6}\left[(3 - x)^3(276) + 0\right]$$

$$+ (3 - x)\left[33 - \frac{1}{6}(276)\right] + (x - 2)[244 - 0]$$

$$= 46(27 - x^3 + 9x^2 - 27x) - 13(3 - x) + 244x - 488$$

$$= -46x^3 + 414x^2 - 985x + 715$$

$\therefore$ Cubic spline is

$$S(x) = \begin{cases} S_1(x) = y_1(x) = -4x^3 + 5x + 1, & 0 \le x \le 1 \\ S_2(x) = y_2(x) = 50x^3 - 162x^2 + 167x - 53, & 1 \le x \le 2 \\ S_3(x) = y_3(x) = -46x^3 + 414x^2 - 985x + 715, & 2 \le x \le 3 \end{cases}$$

When $x = 0.5, y_1(x = 0.5) = S_1(x = 0.5) = -4(0.5)^3 + 5(0.5)^2 + 1 = 3$

When $x = 1.5, y_2(x = 1.5) = S_2(x = 1.5) = 50(1.5)^3 - 162(1.5)^2 + 167(1.5) - 53 = 1.75$

When $x = 2.5, y_3(x = 2.5) = S_3(x = 2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715 = 121.25$

Example 4.11. **From the following table**

x	1	2	3
$y = f(x)$	−8	−1	18

. Find cubic spline and compute $y(1.5), y'(1), y(2.5)$ and $y'(3)$.

Solution:

$$S(x) = \begin{cases} S_1(x) = y_1(x) = 3(x - 1)^3 + 4x - 12, & 1 \le x \le 2 \\ S_2(x) = y_2(x) = 3(3 - x)^3 + 22x - 48, & 2 \le x \le 3 \end{cases}$$

&

$$y(x = 1.5) = S_1(x = 1.5) = -\frac{45}{8}, y'(x = 1) = S_1'(x = 1) = 4$$

$$y(x = 2.5) = S_2(x = 2.5) = 7.375, y'(x = 3) = S_2'(x = 3) = 22.$$

Example 4.12. **Fit a natural cubic spline for the following data:**

x	0	1	2	3
$y = f(x)$	1	4	0	−2

{AU 2008}

Solution: Assume $M_0 = 0 = M_3$.

$$S(x) = \begin{cases} S_1(x) = y_1(x) = -2x^3 + 5x + 1, & [0, 1] \\ S_2(x) = y_2(x) = 3x^3 - 15x^2 + 20x - 4, & [1, 2] \\ S_3(x) = y_3(x) = -x^3 + 9x^2 - 28x + 28, & [2, 3] \end{cases}$$

4.4.3 Anna University Questions

1. If $f(0) = 1, f(1) = 2, f(2) = 33$ and $f(3) = 244$, find a cubic spline approximation, assuming $M(0) = M(3) = 0$. Also, find $f(2.5)$.   (AM10)

    Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -4x^3 + 5x + 1, & x \in [0, 1] \\ S_2(x) = y_2(x) = 50x^3 - 162x^2 + 1670x - 53, & x \in [1, 2] \\ S_3(x) = y_3(x) = -46x^3 + 414x^2 - 985x + 715, & x \in [2, 3] \end{cases}$$

$$f(x) = -46x^3 + 414x^2 - 985x + 715 \quad x \in [2, 3]$$

$$f'(x) = -138x^2 + 828x - 985$$

$$f'(x = 2.5) = -138(2.5)^2 + 828(2.5) - 985$$

$$= 222.5$$

2. Find the natural cubic spline approximation for the function $f(x)$ defined by the following data:

$$x: \quad 0 \quad 1 \quad 2 \quad 3$$
$$f(x): \quad 1 \quad 2 \quad 33 \quad 244$$

(ND10)

Solution: Hint : Ref : Previous AU 1 (AM10)

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -4x^3 + 5x + 1, & x \in [0, 1] \\ S_2(x) = y_2(x) = 50x^3 - 162x^2 + 1670x - 53, & x \in [1, 2] \\ S_3(x) = y_3(x) = -46x^3 + 414x^2 - 985x + 715, & x \in [2, 3] \end{cases}$$

$$\left[f(x) = -46x^3 + 414x^2 - 985x + 715 \Rightarrow f(x = 2.5) = y(x = 2.5) = 121.25, \because x = 2.5 \in [2, 3] \right]$$

3. Find the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0' = y_3' = 0$. (AM11)

$$x: \quad -1 \quad 0 \quad 1 \quad 2$$
$$y: \quad -1 \quad 1 \quad 3 \quad 35$$

Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -2x^3 - 6x^2 - 2x + 1, & x \in [-1, 0] \\ S_2(x) = y_2(x) = 10x^3 - 6x^2 - 2x + 1, & x \in [0, 1] \\ S_3(x) = y_3(x) = -8x^3 + 48x^2 - 56x + 19, & x \in [1, 2] \end{cases}$$

4. The following values of x and y are given:

$$x: \quad 1 \quad 2 \quad 3 \quad 4$$
$$y: \quad 1 \quad 2 \quad 5 \quad 11$$

Find the cubic splines and evaluate $y(1.5)$ and $y'(3)$ (MJ12)

Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right), & x \in [1, 2] \\ S_2(x) = y_2(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right), & x \in [2, 3] \\ S_3(x) = y_3(x) = \frac{1}{3}\left(-2x^3 + 24x^2 - 76x + 81\right), & x \in [3, 4] \end{cases}$$

$$y(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right) \Rightarrow y(1.5) = 1.375, \qquad x \in [1, 2]$$

$$y'(x) = \frac{1}{3}\left(3x^2 - 6x + 5\right) \Rightarrow y'(3) = 4.666666667, \qquad x \in [2, 3]$$

(or)

$$y'(x) = \frac{1}{3}\left(-6x^2 + 48x - 76\right) \Rightarrow y'(3) = 4.666666667, \quad x \in [3, 4]$$

5. Obtain the cubic spline for the following data to find $y(0.5)$.

$$x: \quad -1 \quad 0 \quad 1 \quad 2$$
$$y: \quad -1 \quad 1 \quad 3 \quad 35$$

(ND12)

Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -2x^3 - 6x^2 - 2x + 1, & x \in [-1, 0] \\ S_2(x) = y_2(x) = 10x^3 - 6x^2 - 2x + 1, & x \in [0, 1] \\ S_3(x) = y_3(x) = -8x^3 + 48x^2 - 56x + 19, & x \in [1, 2] \end{cases}$$

6. Using cubic spline, compute $y(1.5)$ from the given data.   (MJ13)

$$x: \quad 1 \quad 2 \quad 3$$
$$y: \quad -8 \quad -1 \quad 18$$

Solution: Hint :

$$S(x) = y(x) = 3x^3 - 9x^2 + 13x - 15$$

$$y(1.5) = 3(1.5)^3 - 9(1.5)^2 + 13(1.5) - 15 = -\frac{45}{8} = -5.625, \quad x \in [1, 2]$$

7. Find the natural cubic spline to fit the data:

$$x: \quad 0 \quad 1 \quad 2$$
$$f(x): \quad -1 \quad 3 \quad 29$$

Hence find $f(0.5)$ and $f(1.5)$.   (ND13)

Solution: Hint :

$$S(x) = y(x) = \frac{11}{2}x^3 - \frac{3}{2}x - 1, \qquad\qquad x \in [0, 1]$$

$$y(0.5) = -1.0625, \qquad\qquad x \in [0, 1]$$

$$S(x) = y(x) = \frac{11}{2}(2 - x)^3 - \frac{5}{2}(2 - x) + 29x - 29, \quad x \in [1, 2]$$

$$y(1.5) = 13.9375, \qquad\qquad x \in [1, 2]$$

8. Fit the cubic splines for the following data.   (16)

     $x: \quad 1 \quad 2 \quad 3 \quad 4 \quad 5$

     $y: \quad 1 \quad 0 \quad 1 \quad 0 \quad 1$ (AM14)

Solution : Assume $M_0 = 0 = M_4$.

$$f(x) = S(x) = \begin{cases} S_1(x) = y_1(x) = 2 - x, & 1 \le x \le 2 \\ S_2(x) = y_2(x) = \dfrac{1}{7}[-5x^3 + 45x^2 - 123x + 106], & 2 \le x \le 3 \\ S_3(x) = y_3(x) = \dfrac{1}{7}[6x^3 - 72x^2 + 275x - 332], & 3 \le x \le 4 \\ S_4(x) = y_4(x) = \dfrac{1}{7}[-5x^3 + 75x^2 - 363x + 772], & 4 \le x \le 5 \end{cases}$$

9. Obtain the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0'' = y_3'' = 0$.   (ND14)

$$x \quad -1 \quad 0 \quad 1 \quad 2$$
$$y \quad -1 \quad 1 \quad 3 \quad 35$$

10. Find the cubic Spline interpolation.   (AU N/D, 2007,AM2014)

x	1	2	3	4	5
$f(x)$	1	0	1	0	1

11. Given the following table, find $f(2.5)$ using cubic spline functions :   (AU May/June 2007)

x	1	2	3	4
$f(x)$	0.5	0.3333	0.25	0.2

Solution :   [Ans: $S_2(2.5) = 0.2829$]

12. Fit the st.line for the data.

x	0	1	2	3
$f(x)$	1	2	9	28

Solution :

$$f(x) = \begin{cases} y_1(x) = \dfrac{4}{5}x^3 - \dfrac{4}{5}x + 1, & 0 \le x \le 1 \\[2mm] y_2(x) = \dfrac{1}{5}[10x^3 - 18x^2 + 19x - 1, & 1 \le x \le 2 \\[2mm] y_3(x) = -2x^3 + \dfrac{102}{5}x^2 - \dfrac{333}{5}x + \dfrac{159}{5}, & 2 \le x \le 3 \end{cases}$$

4.5 Newton's forward and backward difference formulae

Newton's forward and backward difference formulae for Uniform (or) equal intervals only.

Newton's forward interpolation difference formula:

[If y (required x near to x_0) =? and use Δ]

$$y(x) = f(x) = f(x_0 + uh)$$

$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

where $u = \dfrac{x - x_0}{h}, h$ = length of interval.

4.6 Newton's backward interpolation difference formula:

[If y (required x near to x_n) =? and use ∇]

$$y(x) = f(x) = f(x_n + vh)$$

$$= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

where $v = \dfrac{x - x_n}{h}, h$ = length of interval.

4.6.1 Part A

1. When to use Newton's forward interpolation and when to use Newton's backward interpolation?

(ND10)

Solution :

Use of Newton's forward interpolation : The formula is used to interpolate the values of y near the begining of the table value and also for extrapolation the values of y short distance ahead (to the left) of y_0.

Use of Newton's backward interpolation : The formula is used to interpolate the values of y near the end of the table value and also for extrapolation the values of y short distance ahead (to the right) of y_0.

2. State Newton's backward difference formula.   (ND12)

   Solution : Newton's backward interpolation difference formula:

   [If y (required x near to x_n) =? and use ∇]

   $$y(x) = f(x) = f(x_n + vh)$$
   $$= y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots$$

   where $v = \dfrac{x - x_n}{h}, h$ = length of interval.

3. State Newton's forward interpolation formula.   (MJ13)

   Solution : Newton's forward interpolation difference formula:

   [If y (required x near to x_0) =? and use Δ]

   $$y(x) = f(x) = f(x_0 + uh)$$
   $$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

   where $u = \dfrac{x - x_0}{h}, h$ = length of interval

4. State Newton's forward difference formula for equal intervals.   (ND13)

   Solution :   [Ref : Part A : 3 (MJ13)]

5. State Newton's backward formula for interpolation.   (ND14,ND12)

6. What advantage has Lagrange's formula over Newton?

   Solution : The forward and backward interpolation formulae of Newton can be used only when the values of the independent variable x are equally spaced can also be used when the differences of the dependent variable y become smaller ultimately. But Lagrange's interpolation formula can be used whether the values of x, the independent variable are equally spaced or not and whether the difference of y become smaller or not.

7. Derive Newton's forward difference formula by using operator method. (or) Derive Gregory - Newton forward difference interpolation formula.

   Solution : $P_n(x) = P_n(x_0 + uh) = E^u P_n(x_0) = E^u y_0$

   $$= (1 + \Delta)^u y_0$$
   $$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$
   where $u = \dfrac{x - x_0}{h}$.

8. Derive Newton's backward difference formula by using operator method.

Solution :

$$P_n(x) = P_n(x_0 + vh) = E^v P_n(x_n) = E^v y_n$$

$$= (1 - \nabla)^{-v} y_n \text{ where } E = (1 - \nabla)^{-1}$$

$$= \left(1 + v\nabla + \frac{v(v+1)}{2!} \nabla^2 + \frac{v(v+1)(v+2)}{3!} \nabla^3 + \cdots \right) y_n$$

$$= y_n + \frac{v}{1!} \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n + \cdots$$

$$\text{where } v = \frac{x - x_n}{h}$$

4.6.2 Part B

Example 4.13.   Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values

x	0	1	2	3
$y = f(x)$	1	2	1	10

.

Evaluate $f(4)$.   **(AU 2000, 2009)**

Solution: WKT, Newton's forward formula to find the polynomial in x.

There are only 4 data given. Hence the polynomial will be degree 3. Newton's forward formula is

$$y(x) = f(x) = f(x_0 + uh)$$

$$= y_0 + \frac{u}{1!} \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \cdots$$

$$\text{where } u = \frac{x - x_0}{h}$$

$$h = \text{length of interval.}$$

Newton's divided difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 1$			
		$2 - 1 = 1$		
$x_1 = 1$	$y_1 = 2$		$-1 - 1 = -2$	
		$1 - 2 = -1$		$10 - (-2) = 12$
$x_2 = 2$	$y_2 = 1$		$9 - (-1) = 10$	
		$10 - 1 = 9$		
$x_3 = 3$	$y_3 = 10$			

$$f(x) = 1 + \frac{x}{1!}(1) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(12)$$
$$= 2x^3 - 7x^2 + 6x + 1$$

When $x = 4$,

$$f(4) = 2 \times 4^3 - 7 \times 4^2 + 6 \times 4 + 1 = 128 - 92 + 24 + 1 = 41$$

Example 4.14.   The population of a city in a census takes once in 10 years is given below. Estimate the population in the year 1955.

Year	1951	1961	1971	1981
Population in lakhs	35	42	58	84

Solution: $y(x = 1955) = f(x = 1955) = 36.784$

Example 4.15.   From the table given below find $\sin 52°$ by using Newton's forward interpolation formula.

x	45°	50°	55°	60°
$y = \sin x$	0.7071	0.7660	0.8192	0.8660

Solution: $y(x = 52) = \sin 52° = 0.788$ approximately.

Example 4.16.   From the data given below find the number of students whose weight is between 60 and 70.

Weight in lbs	0-40	40-60	60-80	80-100	100-120
Number of students	250	120	100	70	50

Solution: Let weight be denoted by x and

Number of students be denoted by y, i.e., $y = f(x)$.

Use Newton's forward formula to find y where x lies between $60 - 70$.

Newton's forward formula is

$$y(x) = f(x) = f(x_0 + uh)$$
$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

We rewrite the table as cumulative table showing the number of students less than x lbs.

x	Below 40	Below 60	Below 80	Below 100	Below 120
y	250	370	470	540	590

Newton's forward difference table is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
		120			
Below 60	370		−20		
		120		−10	
Below 80	470		−30		20
		70		10	
Below 100	540		−20		
		50			
Below 120	590				

Here $u = \dfrac{x - x_0}{h} = \dfrac{x - 40}{20}$.

When $x = 70, u = \dfrac{70 - 40}{20} = 1.5$

$$y(x = 70) = 250 + \frac{1.5}{1!}(120) + \frac{1.5(1.5 - 1)}{2!}(-20) + \frac{1.5(1.5 - 1)(1.5 - 2)}{3!}(-10)$$
$$+ \frac{1.5(1.5 - 1)(1.5 - 2)(1.5 - 3)}{4!}(20)$$
$$= 423.59 = 424$$

∴ Number of students whose weight is 70 = 424

∴ Number of students whose weight is between $60 - 70 = 424 - 370 = 54$.

Example 4.17. Use Newton's backward difference formula to construct as interpolating polynomial of degree 3 for the data.

$f(0.75) = 0.07181250, f(0.5) = 0.024750, f(0.25) = 0.33493750, f(0) = 1.10100.$ **Find** $f\left(-\dfrac{1}{3}\right)$.

Solution: $y(x) = x^3 + 4.001x^2 + 4.002x + 1.101,\ f\left(-\dfrac{1}{3}\right) = 0.174518518.$

Example 4.18. From the following data, find θ at $x = 43$ and $x = 84$.

x	40	50	60	70	80	90
θ	184	204	226	250	276	304

. Also express θ in terms of x.

Solution: $\theta(x = 43) = 189.79$ (by Newton's forward formula)

$\theta(x = 84) = 286.96$ (by Newton's backward formula)

$\theta(x) = 0.01x^2 + 1.1x + 124$ (by Newton's forward formula)

Example 4.19. Find a polynomial of degree two for the data by Newton's forward difference method :

x	0	1	2	3	4	5	6	7
$f(x)$	1	2	4	7	11	16	22	29

(AU May/June, 2007)

Solution:

$$\left[\text{Ans} : y(x) = \frac{1}{2}\left(x^2 + x + 2\right)\right]$$

4.6.3 Anna University Questions

1. Find the value of $\tan 45°15'$ by using Newton's forward difference interpolation formula for

$x°$	45	46	47	48	49	50
$\tan x°$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

(ND10)

   Solution :   [Ans: $\tan 45°15' = 1.00876$, by Newton's forward difference formula]

2. Derive Newton's backward difference formula by using operator method.   (MJ12)

3. Find the value of y when $x = 5$ using Newton's interpolation formular from the following table:

$$x : \quad 4 \quad 6 \quad 8 \quad 10$$
$$y : \quad 1 \quad 3 \quad 8 \quad 16$$

(ND12)

   Solution :   [$y(x = 5) = 1.625$, by Newton's forward difference formula]

4. Fit a polynomial, by using Newton's forward interpolation formula, to the data given below.   (8)

$$x : \quad 0 \quad 1 \quad 2 \quad 3$$
$$y : \quad 1 \quad 2 \quad 1 \quad 10$$

(MJ13)

   Solution :

$$y(x) = 2x^3 - 7x^2 + 6x + 1, \quad \text{by Newton's forward \& backward formula}$$
$$y(x = 4) = 41$$

5. Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values: (ND14)

x	0	1	2	3
$f(x)$	1	2	1	10

4.6.4 Assignment problems

1. Use Lagrange's formula to find a polynomial which takes the values $f(0) = -12$, $f(1) = 0$, $f(3) = 6$ and $f(4) = 12$. Hence find $f(2)$. **(AM10)**

2. Find the value of x when $y = 20$ using Lagrange's formula from the following table. **(AM11)**

$x :$	1	2	3	4
$y = f(x) :$	1	8	27	64

3. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data:

Year:	1997	1999	2001	2002
Profit in Lakhs Rs. :	43	65	159	248

   (MJ12)

4. Use Lagrange's method to find $\log_{10} 656$, given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, and $\log_{10} 661 = 2.8202$. **(ND12)**

5. Find the function $f(x)$ from the following table using Newton's divided difference formula:

$x :$	0	1	2	4	5	7
$f(x) :$	0	0	-12	0	600	7308

   (AM10)

6. Given tha table

$x :$	5	7	11	13	17
$f(x) :$	150	392	1452	2366	5202

   Evaluate $f(9)$ using Newton's divided difference formula. **(AM11)**

7. Find $f(3)$ by Newton's divided difference formula for the following data: **(AM14, ND2004)**

$x :$	-4	-1	1	2	5
$y :$	1245	33	5	9	1335

8. By using Newton's divided difference formula find $f(8)$, gives **(ND14)**

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

9. If $f(0) = 1$, $f(1) = 2$, $f(2) = 33$ and $f(3) = 244$, find a cubic spline approximation, assuming $M(0) = M(3) = 0$. Also, find $f(2.5)$. **(AM10)**

10. Find the natural cubic spline to fit the data:

$x :$	0	1	2
$f(x) :$	-1	3	29

    Hence find $f(0.5)$ and $f(1.5)$. **(ND13)**

11. Fit the cubic splines for the following data.   (AM14)

     x :   1   2   3   4   5

     y :   1   0   1   0   1

12. Obtain the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0'' = y_3'' = 0$.   (ND14)

x	−1	0	1	2
y	−1	1	3	35

13. Find the value of $\tan 45° 15'$ by using Newton's forward difference interpolation formula for

$x°$	45	46	47	48	49	50
$\tan x°$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

(ND10)

14. Find the value of y when $x = 5$ using Newton's interpolation formular from the following table:

     x :   4   6   8   10

     y :   1   3   8   16   (ND12)

15. Fit a polynomial, by using Newton's forward interpolation formula, to the data given below.

     x :   0   1   2   3

     y :   1   2   1   10   (MJ13)

16. Using Newton's forward interpolation formula, find the cubic polynomial which takes the following values: (ND14)

x	0	1	2	3
$f(x)$	1	2	1	10

4.7 Numerical Differentiation and Integration

Numerical Differentiation	Numerical Integration
Newton's forward difference formula to compute derivatives (Equal interval)	Trapezoidal rule [$n = 1$ in Quadrature formula]
Newton's backward difference formula to compute derivatives(Equal interval)	Simpson's one third rule [$n = 2$ in Quadrature formula]
Lagrange's Interpolation formula (Equal or unequal intervals)	Simpson's three eighth rule [$n = 3$ in Quadrature formula]
Newton's divided difference Interpolation formula (Equal or unequal intervals)	Romberg's method
Maxima and minima of a tabulated function	Two point Gaussian's quadrature formula
	Three point Gaussian's quadrature formula
	Double integrals by Trapezoidal rule
	Double integrals by Simpson's 1/3 rule

4.8 Approximation of derivatives using interpolation polynomials

Numerical Differentiation

Given $y = y(x) = f(x)$ [in a table]

$\dfrac{dy}{dx} = y'(x) = f'(x)$ is the first numerical derivative

$\dfrac{d^2y}{dx^2} = y''(x) = f''(x)$ is the second numerical derivative

$\dfrac{d^ny}{dx^n} = y^{1\cdots n}(x) = f^{1\cdots n}(x)$ is the nth numerical derivative

4.8.1 Newton's forward difference formula to compute derivative

WKT, Newton's forward difference interpolation formula is

$$y(x) = f(x_0+uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \frac{u^4-6u^3+11u^2-6u}{4!}\Delta^4 y_0 + \cdots$$

$$\text{where } u = \frac{x-x_0}{h}$$

First derivative

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{dy}{du}$$

$$\text{i.e., } \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{\substack{\text{at } x = x_0 \\ \Rightarrow u = 0}} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{du}\frac{du}{dx}\right] = \frac{d}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{du}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{1}{h}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h}\frac{d}{du}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h^2}\frac{d^2y}{du^2}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{12u^2 - 36u + 22}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]\genfrac{[}{]}{0pt}{}{}{\begin{array}{l}\text{at } x = x_0 \\ \Rightarrow u = 0\end{array}} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{22}{24}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

Third derivative

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \left(\frac{24u - 36}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]\genfrac{[}{]}{0pt}{}{}{\begin{array}{l}\text{at } x = x_0 \\ \Rightarrow u = 0\end{array}} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{36}{24}\Delta^4 y_0 + \cdots\right]$$

4.8.2 Newton's backward difference formula to compute derivatives

WKT, Newton's backward difference interpolation formula is

$$y(x) = f(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

$$= y_n + \frac{v}{1!}\nabla y_n + \left(\frac{v^2 + v}{2!}\right)\nabla^2 y_n + \left(\frac{v^3 + 3v^2 + 2v}{3!}\right)\nabla^3 y_n + \left(\frac{v^4 + 6v^3 + 11v^2 + 6v}{4!}\right)\nabla^4 y_n + \cdots$$

where $v = \dfrac{x - x_n}{h}$

First derivative

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{dy}{dv}$$

$$= \frac{1}{h}\left[\nabla y_n + \left(\frac{2v + 1}{2}\right)\nabla^2 y_n + \left(\frac{3v^2 + 6v + 2}{3!}\right)\nabla^3 y_n + \left(\frac{4v^3 + 18v^2 + 22v + 6}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]\genfrac{[}{]}{0pt}{}{}{\begin{array}{l}\text{at } x = x_n \\ \Rightarrow v = 0\end{array}} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{dv}\frac{dv}{dx}\right] = \frac{d}{dx}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{d}{dv}\frac{dv}{dx}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{d}{dv}\frac{1}{h}\left[\frac{1}{h}\frac{dy}{dv}\right] = \frac{1}{h^2}\left[\frac{d^2y}{dv^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \left(\frac{6v + 6}{3!}\right)\nabla^3 y_n + \left(\frac{12v^2 + 36v + 22}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]\genfrac{[}{]}{0pt}{}{}{\begin{array}{l}\text{at } x = x_n \\ \Rightarrow v = 0\end{array}} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6}{3!}\nabla^3 y_n + \frac{22}{4!}\nabla^4 y_n + \cdots\right]$$

<u>Third derivative</u>

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \left(\frac{24v + 36}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]\left[\begin{array}{c} \text{at } x = x_n \\ \Rightarrow v = 0 \end{array}\right] = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \frac{36}{4!}\nabla^4 y_n + \cdots\right]$$

4.8.3 Maxima and Minima of a tabulated function

(If the intervals are same)

WKT, Newton's forward difference interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

<u>First derivative</u>

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \cdots\right] \tag{1}$$

substitute $h, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \cdots$ gives the equation

$$\frac{dy}{dx} = \text{ an equation in } u \tag{2}$$

For maxima or minima is obtained by the equation by

$$\frac{dy}{dx} = 0 \text{ (or) } y'(x) = 0$$

Put RHS of (1) = 0, find $u = u_1, u_2, \cdots$

Find $\dfrac{d^2y}{dx^2}$ from (2):

Suppose $\left(\dfrac{d^2y}{dx^2}\right)_{\text{at } u_1} = -\text{ve} \Rightarrow u_1$ is maximum point

$$\Rightarrow y(u_1) = \text{ maximum value of } y$$

Suppose $\left(\dfrac{d^2y}{dx^2}\right)_{\text{at } u_2} = +\text{ve} \Rightarrow u_2$ is minimum point

$$\Rightarrow y(u_2) = \text{ minimum value of } y$$

4.8.4 Part A

1. Write down the formulae for finding the first derivative using Newton's forward difference at $x = x_0$ Newton's backward difference at $x = x_n$.   (ND2010)

   Solution : First derivative by Newton's forward difference interpolation formula is

   $$\left[\frac{dy}{dx}\right]\left[\begin{array}{c} \text{at } x = x_0 \\ \Rightarrow u = 0 \end{array}\right] = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

First derivative by Newton's backward difference interpolation formula is

$$\left[\frac{dy}{dx}\right]_{\substack{\text{at } x = x_n \\ \Rightarrow v = 0}} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

2. Write down the expression for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_n$ by Newton's backward difference formula.

(AM2014)

First derivative by Newton's backward difference interpolation formula is

$$\left[\frac{dy}{dx}\right]_{\substack{\text{at } x = x_n \\ \Rightarrow v = 0}} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

Second derivative by Newton's backward difference interpolation formula is

$$\left[\frac{d^2y}{dx^2}\right]_{\substack{\text{at } x = x_n \\ \Rightarrow v = 0}} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6}{3!}\nabla^3 y_n + \frac{22}{4!}\nabla^4 y_n + \cdots\right]$$

4.8.5 Part B

Example 4.20. **The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds.**

Time(sec)	0	5	10	15	20
Velocity (m/sec)	0	3	14	69	228

Find (a) Initial acceleration using the entire data   (b) Final acceleration.

Solution: The difference table is

Time $t = x$	Velocity $v = y(x)$	$\Delta y(x)$	$\Delta^2 y(x)$	$\Delta^3 y(x)$	$\Delta^4 y(x)$
$0 (= x_0)$	$0 (= y_0)$				
		$3 (= \Delta y_0)$			
5	3		$8 \left(= \Delta^2 y_0\right)$		
		11		$36 \left(= \Delta^3 y_0\right)$	
10	14		44		$24 \left(= \Delta^4 y_0 \text{ or } \nabla^4 y_n\right)$
		55		$60 \left(= \nabla^3 y_n\right)$	
15	69		$104 \left(= \nabla^2 y_n\right)$		
		$159 (= \nabla y_n)$			
20	$228 (= y_n)$				

Here $x_0 = 0, h = $ interval length $= 5$

(a) WKT, acceleration $= \frac{dv}{dt} = $ rate of change of velocity

To find initial acceleration, put $\left(\frac{dv}{dt}\right)_{t=t_0} = \left(\frac{dv}{dt}\right)_{t=0}$.

i.e., initial acceleration exists at $t = 0 = x_0$ [which is nearer to beginning of the table], so we use

Newton's forward difference formula for first derivative.

∴ Newton's forward difference interpolation formula is

$$y(x) = f(x_0 + uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Now, Newton's forward difference formula for first derivative at $x = x_0 = 0$ [$\Rightarrow u = 0$]

i.e., $y'(x_0) = \left(\dfrac{dv}{dt}\right)_{t=0} = \dfrac{1}{h}\left[\Delta y_0 - \dfrac{1}{2}\Delta^2 y_0 + \dfrac{1}{3}\Delta^3 y_0 - \dfrac{1}{4}\Delta^4 y_0 + \cdots\right]$ 　　　　[Here $h = 5$]

$$= \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right] = 1$$

$\Rightarrow y'(0) = 1$

∴ Initial acceleration (acceleration when $t = 0$) is 1 m/sec^2.

(b) Final acceleration exists at $t = 20 = x_4$ [Nearer to ending of table], so use Newton's backward difference interpolation formula for first derivative, put $\left(\dfrac{dv}{dt}\right)_{t=t_n} = \left(\dfrac{dv}{dt}\right)_{t=20}$.

WKT, Newton's backward difference interpolation formula is

$$y(x) = f(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

$$+ \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

$$y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{5}\left[159 + \frac{1}{2}(104) + \frac{2}{6}(60) + \frac{6}{24}(24)\right]$$

$$= \frac{1}{5}(237) = 47.2 \text{ m/sec}^2$$

Example 4.21.　Consider the following table of data :

x	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.9798652	0.9177710	0.8080348	0.636093	0.3843735

Find (1) $f'(0.25)$ using Newton's forward difference approximation

(2) $f'(0.95)$ using Newton's backward difference approximation

Solution: Here $h = 0.2$

(1) $u = \dfrac{x - x_0}{h} = \dfrac{x - 0.2}{0.2} = \dfrac{0.25 - 0.2}{0.2} = 0.25$

∴$f'(0.25) = -0.25828$

(2) $u = \dfrac{x - x_n}{h} = \dfrac{0.95 - 1}{0.2} = -0.25$

$f'(0.95) = -1.367948$

Example 4.22.　Find the value of $f'(8), f''(9)$, maximum and minimum value from the following data, using an approximate interpolation formula.

x	4	5	7	10	11
$f(x)$	48	100	294	900	1210

Solution: The values of x are unequally spaced.

To find $f(x)$, we use Newton's divided difference formula (or) Lagrange formula.

WKT, Newton's Divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3)$$

$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \cdots \qquad (1)$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		
		310			
11	1210				

$$\therefore (1) \Rightarrow f(x) = 48 + (x - 4)(52) + (x - 4)(x - 5)(15) + (x - 4)(x - 5)(x - 7)(1)$$

$$= 48 + 52x - 208 + 15\left[x^2 - 9x + 20\right] + x^3 - x^2(16) + x(83) - 140$$

$$= x^3 + x^2(-16 + 15) + x(83 - 135 + 52) - 208 + 300 - 140 + 48$$

$$= x^3 - x^2 + x(0) + 0$$

$$\therefore f(x) = x^3 - x^2$$

$$f'(x) = 3x^2 - 2x, \quad f'(8) = 176$$

$$f''(x) = 6x - 2, \quad f''(9) = 52$$

<u>To find maximum and minimum</u>

$$\text{Put } f'(x) = 0 \Rightarrow 3x^2 - 2x = 0 \Rightarrow x = 0, x = \frac{2}{3}$$

At $x = 0$, $f''(x = 0) = 6(0) - 2 = -2 < 0$

$\Rightarrow x = 0$ is a maximum point & maximum value is $f(x = 0) = 0$.

At $x = \frac{2}{3}$, $f''\left(x = \frac{2}{3}\right) = 6\left(\frac{2}{3}\right) - 2 = 2 > 0$.

$\Rightarrow x = \frac{2}{3}$ is a minimum point & minimum value is $f\left(x = \frac{2}{3}\right) = -\frac{4}{27}$

Example 4.23. **Evaluate y' and y'' at $x = 2$ given**

x	0	1	3	6
y	18	10	−18	40

Solution:

$$\left[\text{Ans}: y(x) = x^3 - \frac{70}{9}x^2 - \frac{15}{9}x + 18, y'(x = 2) = -\frac{187}{9}, y''(x = 2) = -\frac{22}{9}\right]$$

Example 4.24.   Find the value of $\cos(1.747)$ using the values given in the table below :

x	1.70	1.74	1.78	1.82	1.86
$\sin x$	0.9916	0.9857	0.9781	0.9691	0.9584

Solution: [Ans : -0.175]

Example 4.25.   Find $\sec 31°$ from the following data :

θ	31	32	33	34
$\tan \theta$	0.6008	0.6249	0.6494	0.6745

Solution: $\left[\text{Ans} : \sec^2 31 = 1.3835 \Rightarrow \sec 31 = 1.174, \text{ Hint} : 1° = \dfrac{\pi}{180} = 0.017453292\right]$

4.8.6 Anna University Questions

1. Given the following data, find $y'(6)$ and the maximum value of y (if it exists).   (AM10)

$$x : \quad 0 \quad 2 \quad 3 \quad 4 \quad 7 \quad 9$$
$$y : \quad 4 \quad 26 \quad 58 \quad 112 \quad 466 \quad 922$$

2. Find $f'(x)$ at $x = 1.5$ and $x = 4.0$ from the following data using Newton's formulae for differentiation.

$$x : \qquad 1.5 \quad 2.0 \quad 2.5 \quad 3.0 \quad 3.5 \quad 4.0$$
$$y = f(x) : \quad 3.375 \quad 7.0 \quad 13.625 \quad 24.0 \quad 38.875 \quad 59.0$$
                                                                                  (MJ12)

3. Find the first three derivatives of $f(x)$ at $x = 1.5$ by using Newton's forward interpolation formula to the data given below.

$$x : \qquad 1.5 \quad 2 \quad 2.5 \quad 3 \quad 3.5 \quad 4$$
$$y : \quad 3.375 \quad 7 \quad 13.625 \quad 24 \quad 38.875 \quad 59$$
                                                                                  (MJ13)

Numerical integration

Newton-Cote's quadrature formula is

$$\int_{a}^{a+nh} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

where n = number of intervals

h = length of interval

4.9 Numerical integration using Trapezoidal

Trapezoidal rule($n = 1$ in quadrature formula)

$$\int_{x_0}^{x_0+nh} f(x)dx = \begin{cases} \dfrac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] \\ \text{(or)} \\ \dfrac{h}{2}\left[(\text{first term} + \text{last term}) + 2\,(\text{remaining terms})\right] \end{cases}$$

4.9.1 Part A

1. Write down the Newton-Cote's formula for the equidistant ordinates.   (MJ2011)

    Solution : The general Newton-Cote's quadrature formula is

    $$\int_{a}^{a+nh} f(x)\,dx = nh\left[y_0 + \frac{n\Delta}{2}y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

    This is known as the general Gauss-Lagendre integration formula. Putting $n = 1$ and omitting second and hyper differences in the above, we get

    $$\int_{a}^{a+h} f(x)\,dx = \frac{h}{2}\left[f(a) + f(a+h)\right] = \frac{h}{2}(y_1 + y_2)$$

    which is the trapezoidal rule.

2. Evaluate $\int_{0}^{1} \frac{dx}{1+x^2}$ using Trapezoidal rule.   (ND2012)

    Solution : Let $y = y(x) = f(x) = \frac{1}{1+x^2}, h = 0.2 = \frac{1}{5}$

x	0	0.2	0.4	0.6	0.8	1
$y = \frac{1}{1+x^2}$	1	0.96154	0.86207	0.73529	0.60976	0.5
	y_0	y_1	y_2	y_3	y_4	y_5

$$\int_{x_0}^{x_0+nh} f(x)dx = \begin{cases} \frac{h}{2}\left[(\text{first term} + \text{last term}) + 2\,(\text{remaining terms})\right] \\ (\text{or}) \\ \frac{h}{2}\left[(y_0 + y_n) + 2\,(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] \end{cases}$$

$$= \frac{h}{2}\left[(y_0 + y_5) + 2\,(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}\left[(1 + 0.5) + 2\,(0.96154 + 0.86207 + 0.73529 + 0.60976)\right]$$

$$\equiv 0.78373$$

3. Evaluate $\int_{0}^{\pi} \sin x\,dx$ by Trapezoidal rule by dividing ten equal parts.   (AM2013)

    Solution :   [1.9843]

4. Evaluate $\int_{0}^{6} \frac{dx}{1+x^2}$ using Trapezoidal rule.   (ND2013)

    Solution :   [1.41079950]

5. Taking $h = 0.5$, evaluate $\int_{1}^{2} \frac{dx}{1+x^2}$ using Trapezoidal rule.   (AM2014)

    Solution :   [0.3289]

4.9.2 Part B

Example 4.26. Using Trapezoidal rule, evaluate $\int_{-1}^{1} \dfrac{1}{(1+x^2)} dx$ **by taking eight equal intervals.**

(MJ13)

Solution: Let $y = y(x) = f(x) = \dfrac{1}{1+x^2}$

Lenth of the given interval $[a, b] = b - a = 1 - (-1) = 2$

Lenth of the 8 equal intervals $= h = \dfrac{2}{8} = 0.25$

Form the table for the ordinates of the function $y = y(x) = f(x) = \dfrac{1}{1+x^2}$

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
$y = \dfrac{1}{1+x^2}$	0.5	0.64	0.8	0.94118	1	0.94118	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$\int_{x_0}^{x_0+nh} f(x)dx = \begin{cases} \dfrac{h}{2}\left[(\text{first term} + \text{last term}) + 2\,(\text{remaining terms})\right] \\ (\text{or}) \\ \dfrac{h}{2}\left[(y_0 + y_n) + 2\,(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] \end{cases}$$

$$= \dfrac{h}{2}\left[(y_0 + y_8) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \dfrac{0.2}{2}\left[(0.5 + 0.5) + 2\,(0.64 + 0.8 + 0.94118 + 1 + 0.94118 + 0.8 + 0.64)\right]$$

$$\equiv 1.56559$$

4.9.3 Anna University Questions

1. Using Trapezoidal rule, evaluate $\int_{-1}^{1} \dfrac{1}{(1+x^2)} dx$ by taking eight equal intervals.   (MJ13)

   Solution: [1.5656]

4.10 Numerical integration using Simpson's 1/3 rule

Simpson's one third rule (Simpson's 1/3 rule) ($n = 2$ in quadrature form)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \dfrac{h}{3}\left[(y_0 + y_n) + 4\,(y_1 + y_3 + y_5 + \cdots) + 2\,(y_2 + y_4 + y_6 + \cdots)\right] \\ (\text{or}) \\ \dfrac{h}{3}\left[\begin{array}{l}(\text{first term} + \text{last term}) \\ +4\,(\text{odd suffices}) \\ +2\,(\text{even suffices})\end{array}\right] \end{cases}$$

4.10.1 Part A

1. When do you apply Simpson's $\dfrac{1}{3}$ rule, and what is the order of the error in Simpson's $\dfrac{1}{3}$ rule.
   (MJ2011)

Solution : Let n = number of intervals

Simpson's $\frac{1}{3}$ rule : The number of ordinates is odd (or) the intervals number is even.

The error in the Simpson's 1/3 rule is of the order h^4.

2. State Simpson's one-third rule.   (ND2011,AM2013)

   Solution : Let DC be the curve $y = f(x)$ and DA, CB be the terminal ordinates. Let $OA = a$ and $OB = b$. Divide AB into even number(say $2n$) of equal parts, equal to h. Let $x_1, x_2, \cdots, x_{2n+1}$ be the abscissae of the points, A, A_1, B and $y_1, y_2, \cdots, y_{2n+1}$ be the corresponding ordinates. Then

   $$\int_a^b y\,dx \text{ is approximately} = \frac{h}{3}(A + 4B + 2C) \tag{1}$$

   where $A = y_1 + y_{2n+1}$ = sum of the first and last ordinates.

   $B = y_2 + y_1 + \cdots + y_{2n}$ = sum of the even ordinates and

   $C = y_3 + y_5 + \cdots + y_{2n-1}$ = sum of the remaining ordinates (1)

   known as Simpson's rule or Simpson's one - third rule.

3. State the local error term in Simpson's $\frac{1}{3}$ rule.   (ND14)

4.10.2 Part B

Example 4.27. **A curve passes through the points $(1, 2)$, $(1.5, 2.4)$, $(2, 2.7)$, $(2.5, 2.8)$, $(3, 3)$, $(3.5, 2.6)$ & $(4, 2.1)$. Obtain Area bounded by the curve, x axis between $x = 1$ and $x = 4$. Also find the volume of solids of revolution by revolving this area about $x-$ axis.**
Solution:

$$\text{WKT, Area } = \int_a^b y\,dx = \int_1^4 y\,dx$$

$$\text{Simpson's 1/3 rule, Area } = \int_1^4 y\,dx = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= 7.783$$

$$\text{Volume } = \pi \int_a^b y^2\,dx = \pi \int_1^4 y^2\,dx$$

To find y^2 :

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1
y^2	4	5.76	7.29	7.84	9	6.76	4.41

$$\therefore \text{ Volume } = \pi \int_1^4 y^2\,dx = \pi \left\{ \frac{h}{3}\left[(y_0^2 + y_6^2) + 4(y_1^2 + y_3^2 + y_5^2) + 2(y_2^2 + y_4^2) \right] \right\} \qquad \text{[by Simpson's 1/3 rule]}$$

$$= 64.07 \text{ cubic units}$$

4.10.3 Anna University Questions

1. The velocity v of a particle at a distance S from a point on its path is given by the table below:

S (meter)	0	10	20	30	40	50	60
v (m / sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpson's $1/3^{rd}$ rule and Simpson's $3/8^{th}$ rule.

(AM10)

Solution: [By S(1/3) : $I = 1.06338$, By S(3/8) : $I = 1.06425$]

4.11 Numerical integration using Simpson's 3/8 rule

Simpson's three eighth rule (Simpson's 3/8 rule) ($n = 3$ in quadratic form)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \dfrac{3h}{8}\begin{bmatrix} (y_0 + y_n) \\ +2(y_3 + y_6 + \cdots) \\ +3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) \end{bmatrix} \\ \text{(or)} \\ \dfrac{3h}{8}\begin{bmatrix} (\text{first term} + \text{last term}) \\ +2(\text{suffices with a multiple of 3}) \\ +3(\text{remaining terms}) \end{bmatrix} \end{cases}$$

4.11.1 Part A

1. Under what condition, Simpson's 3/8 rule can be applied and state the formula. (AM2012)

   Solution: Condition : The number of intervals should be multiple of three.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \dfrac{3h}{8}\begin{bmatrix} (y_0 + y_n) \\ +2(y_3 + y_6 + \cdots) \\ +3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) \end{bmatrix} \\ \text{(or)} \\ \dfrac{3h}{8}\begin{bmatrix} (\text{first term} + \text{last term}) \\ +2(\text{suffices with a multiple of 3}) \\ +3(\text{remaining terms}) \end{bmatrix} \end{cases}$$

4.11.2 Part B

Example 4.28. **Compute the value of** $\int_{1}^{2} \dfrac{dx}{x}$ **using Simpson's 3/8 rule**

Solution: Let $y = f(x) = \dfrac{1}{x}$,

$$h = 1/3$$

The tabulated values of $y = f(x)$ are

	x_0	x_1	x_2	x_3
x	1	4/3	5/3	6/3 = 2
$f(x)$	$\frac{1}{1} = 1$	$\frac{1}{4/3} = 3/4 = .75$	$\frac{1}{5/3} = 0.6$	$\frac{1}{2} = 0.5$
	y_0	y_1	y_2	y_3

WKT, Simpson's 3/8 rule is

$$\int_a^b f(x)\, dx = \frac{3h}{8} \left[(y_0 + y_4) + 3(y_1 + y_2) + 2(y_3) \right]$$

i.e., $\int_1^2 \left(\frac{1}{x} \right) dx = \frac{3(1/3)}{8} \left[(1 + 0.5) + 3(0.75 + 0.6) + 2(0) \right]$

$$= 0.69375$$

By actual integration,

$$\int_1^2 \frac{1}{x}\, dx = (\log_e x)_1^2 = (\ln 2 - \ln 1) = 0.69315$$

4.11.3 Single integrals by Trapezoidal, Simpson 1/3 & 3/8

Example 4.29. Compute the value of $\int_1^2 \frac{dx}{x}$ using
(a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule

Solution: Here $h = 0.25$,

$$y = f(x) = \frac{1}{x}$$

The tabulated values of $y = f(x)$ are

	x_0	x_1	x_2	x_3	x_4
x	1	1.25	1.50	1.75	2
$f(x)$	$\frac{1}{1} = 1$	$\frac{1}{1.25} = 0.8$	$\frac{1}{1.5} = 0.66667$	$\frac{1}{1.75} = 0.57143$	$\frac{1}{1.2} = 0.5$
	y_0	y_1	y_2	y_3	y_4

(a) WKT, Trapezoidal rule is

$$\int_a^b f(x)\, dx = \int_{x_0}^{x_0 + nh} f(x)\, dx = \frac{h}{2} \left[(y_0 + y_4) + 2(y_1 + y_2 + y_3) \right]$$

$$\Rightarrow \int_1^2 \left(\frac{1}{x} \right) dx = \frac{0.25}{2} \left[(1 + 0.5) + 2(0.8 + 0.66667 + 0.57143) \right]$$

$$= \frac{0.25}{2} [1.5 + 4.0762] = \frac{0.25}{2} [5.5762]$$

$$= 0.697025$$

(b) WKT, Simpson's 1/3 rule is

$$\int_a^b f(x)\, dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3) + 2(y_2)\right]$$

$$\Rightarrow \int_1^2 \left(\frac{1}{x}\right) dx = \frac{0.25}{3}\left[(1 + 0.5) + 4(0.8 + 0.57143) + 2(0.66667)\right]$$

$$= 0.693255$$

(c) WKT, Simpson's 3/8 rule is

$$\int_a^b f(x)\, dx = \frac{3h}{8}\left[(y_0 + y_4) + 3(y_1 + y_2) + 2(y_3)\right]$$

$$\text{i.e., } \int_1^2 \left(\frac{1}{x}\right) dx = \frac{3(0.25)}{8}\left[(1 + 0.5) + 3(0.8 + 0.66667) + 2(0.57143)\right]$$

$$= 0.66024$$

By actual integration,

$$\int_1^2 \frac{1}{x}\, dx = (\log_e x)_1^2 = (\ln 2 - \ln 1) = 0.69315$$

Example 4.30. **Evaluate** $\int_0^{10} \dfrac{dx}{1 + x}$ **by dividing the range into 8 equal parts by (a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule**

Solution: Here $h = 1.25$, $y = f(x) = \dfrac{1}{1 + x}$

(a) I = 2.51368 , (b) I = 2.42200 , (c) I = 2.41838 , Actual integration, $\int_0^{10} \dfrac{dx}{1 + x} = 2.39790$.

Example 4.31. **Evaluate** $\int_0^1 x\, e^x dx$ **taking 4 equal intervals by (a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule**

Solution: (a) 1.02307 , (b) 1.00017 , (c) 0.87468 , AI = 1

Example 4.32. **Calculate** $\int_0^{\pi} \sin^3 x\, dx$ **taking 7 ordinates (6 intervals) using a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule**

Solution: (a)1.33467 , (b)1.32612 , (c)1.30516 , AI = 1.3333.

4.11.4 Anna University Questions

1. The velocity v of a particle at a distance S from a point on its path is given by the table below:

S (meter)	0	10	20	30	40	50	60
v (m / sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpson's $1/3^{rd}$ rule and Simpson's $3/8^{th}$ rule.

(AM10)

Solution: [S(1/3) : I = 1.06338, S(3/8) : I = 1.06425]

2. Evaluate $I = \int\limits_{0}^{6} \dfrac{1}{1+x} dx$ by using (i) direct integration (ii) Trapezoidal rule (iii) Simpson's one-third rule (iv) Simpson's three-eighth rule.   (ND11)

Solution: [By Dir. Int. : I = 1.9459, Trap. : I = 2.022, S(1/3) : I = 1.9587, S(3/8) : I = 1.966]

3. Compute $\int\limits_{0}^{\pi/2} \sin x\, dx$ using Simpson's 3/8 rule.   (ND12)

Solution: [S(3/8) : I = .9999988 $\approx$ 1]

4. The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time(min):	0	2	4	6	8	10	12
Velocity(km/hr):	0	22	30	27	18	7	0

Using Simpson's $\dfrac{1}{3}$–rd rule find the distance covered by the car.   (ND13)

Solution: [By S(1/3) rule, the distance covered by the car : I = 3.55 km,]

5. Taking $h = 0.05$ evaluate $\int\limits_{1}^{1.3} \sqrt{x}\, dx$ using Trapezoidal rule and Simpson's three-eighth rule.(AM14)

Solution: [By Trap. : I = 0.32147, S(3/8) : I = 0.321485354]

6. The velocity v of a particle at a distance α from a point on its path is given by the table: (ND14)

$s(ft)$	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet by using Simpson's $\dfrac{3}{8}$ rule.

4.12 Romberg's method

Romberg's method for a given interval $\left(I = \int\limits_{a}^{b} f(x)dx \right)$

when $h = \dfrac{b-a}{2}$, by trapezoidal rule, we get I_1

when $h = \dfrac{b-a}{4}$, by trapezoidal rule, we get I_2

when $h = \dfrac{b-a}{8}$, by trapezoidal rule, we get I_3

$$\text{Romberg's formula for } I_1 \& I_2 = I_{RM_{1,2}} = I_2 + \frac{(I_2 - I_1)}{3}$$

$$\text{Romberg's formula for } I_2 \& I_3 = I_{RM_{2,3}} = I_3 + \frac{(I_3 - I_2)}{3}$$

$$\text{If } I_{RM_{1,2}} = I_{RM_{2,3}}, \text{ then we can equal } I = I_{RM_{1,2}} = I_{RM_{2,3}}$$

Note : Check, use actual integration, we get $I_{AI} = I = I_{RM_{1,2}} = I_{RM_{2,3}}$

4.12.1 Part A

1. State the Romberg's integration formula with h_1 and h_2. Further, obtain the formula when $h_1 = h$ and $h_2 = \dfrac{h}{2}$. (MJ2010)

   Solution : $I = \dfrac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}$, where I_1 the value of the integral with h_1

   I_2 the value of the integral with h_2

   If $h_1 = h$ & $h_2 = \dfrac{h}{2}$ we get

   $$I = \frac{4I_2 - I_1}{3} = I_2 + \frac{1}{3}(I_2 - I_1)$$

2. State Romberg's integration formula to find the value of $I = \int\limits_a^b f(x)dx$ for first two intervals.(ND14)

4.12.2 Part B

Example 4.33. **Evaluate** $\int\limits_0^2 \dfrac{dx}{x^2 + 4}$ **using Romberg's method. Hence obtain an approximate value for π.**

Solution: To find I_1

When $h = \dfrac{2 - 0}{2} = 1, y = f(x) = \dfrac{1}{x^2 + 4}$

Let $I = \int\limits_0^2 \dfrac{dx}{x^2 + 4}$

The tabulated values of y are

x	0	1	2
$f(x) = \dfrac{1}{x^2 + 4}$	$\dfrac{1}{0^2 + 4} = 0.25$	0.2	0.125
	y_0	y_1	y_2

Using Trapezoidal rule,

$$I_1 = \int\limits_0^2 \frac{dx}{x^2 + 4} = \frac{h}{2}\left[(y_0 + y_2) + 2(y_1)\right]$$

$$= \frac{1}{2}\left[(0.25 + 0.125) + 2(0.2)\right]$$

$$= 0.3875$$

To find I_2

$h = \dfrac{2 - 0}{4} = \dfrac{1}{2} = 0.5$

The tabulated values of y are

x	0	0.5	1	1.5	2
$f(x) = \dfrac{1}{x^2 + 4}$	0.25	0.23529	0.2	0.160	0.125
	y_0	y_1	y_2	y_3	y_4

Using Trapezoidal rule,

$$I_2 = \int_0^2 \frac{dx}{x^2 + 4} = \frac{h}{2} [(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= 0.25 [(0.25 + 0.125) + 2(0.23529 + 0.2 + 0.160)]$$

$$= 0.39136$$

To find I_3

$$h = \frac{2-0}{8} = \frac{1}{4} = 0.25$$

The tabulated values of y are

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$f(x) = \dfrac{1}{x^2+4}$	0.25	0.24615	0.23529	0.21918	0.2	0.17918	0.160	0.14159	0.125
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Trapezoidal rule,

$$I_3 = \int_0^2 \frac{dx}{x^2 + 4} = \frac{h}{2} [(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= 0.125 \Big[(0.25 + 0.125) + 2(0.24615 + 0.23529 + 0.21918 + 0.2 + 0.17918 + 0.16 + 0.14159) \Big]$$

$$= 0.39237$$

Romberg's formula for $I_1 \& I_2$ is

$$I_{RM_{1,2}} = I_2 + \frac{(I_2 - I_1)}{3} = 0.39136 + \frac{(0.39136 - 0.39136)}{3}$$

$$= 0.39265$$

Romberg's formula for $I_2 \& I_3$ is

$$I_{RM_{2,3}} = I_3 + \frac{(I_3 - I_2)}{3} = 0.39237 + \frac{(0.39237 - 0.39237)}{3}$$

$$= 0.39271$$

$$\therefore I_{RM_{1,2}} \cong 0.3927$$

$$I_{RM_{2,3}} \cong 0.3927$$

Here $I_{RM_{1,2}} \& I_{RM_{2,3}}$ are almost equal and $I = 0.3927$ (1)

By actual integration,

$$\int_0^2 \frac{dx}{x^2 + 4} = \frac{1}{2} \left[\tan^{-1} \frac{x}{2} \right]_{x=0}^{x=2} = \frac{1}{2} \left[\tan^{-1}(1) - \tan^{-1}(0) \right]$$

$$= \frac{1}{2} \left[\frac{\pi}{4} - 0 \right] = \frac{\pi}{8}$$ (2)

From (1) & (2),

$$\frac{\pi}{8} = 0.3927$$

$$\Rightarrow \pi = 8(0.3927) = 3.1416$$

Example 4.34. Using Romberg's method, evaluate $\int\limits_{0}^{1} \dfrac{1}{1+x} dx$ correct to 3 places of decimals.

Solution: $I_1 = 0.7083$

$$I_2 = 0.6970$$

$$I_3 = 0.6941$$

$$I_{RM_{1,2}} = 0.6932$$

$$I_{RM_{2,3}} = 0.6931$$

$$\therefore I = 0.693$$

4.12.3 Anna University Questions

1. Using Romberg's integration to evaluate $\int\limits_{0}^{1} \dfrac{dx}{1+x^2}$.   (AM10)

   Solution: [0.7854]

2. Using Romberg's rule evaluate $\int\limits_{0}^{1} \dfrac{1}{1+x} dx$ correct to three decimal places by taking $h = 0.5, 0.25,$ and 0.125.   (ND10)

   Solution: [0.6931]

3. Evaluate $\int\limits_{0}^{1} \dfrac{1}{1+x^2} dx$ using Romberg's method.   (AM11,AM10)

   Solution: [0.7854]

4. Use Romberg's method to compute $\int\limits_{0}^{1} \dfrac{1}{1+x^2} dx$ correct to 4 decimal places. Also evaluate the same integral using tree-point Gaussian quadrature formula. Comment on the obtained values by comparing with the exact value of the integral which is equal to $\dfrac{\pi}{4}$.   (MJ12)

   Solution: [By Rom. : $I = 0.7854$, Dir. Int. : $I = 0.7853982$]

5. Evaluate $\int\limits_{0}^{\frac{1}{2}} \dfrac{x}{\sin x} dx$ correct to three decimal places using Romberg's method.   (AM14)

   Solution: [0.5070676]

6. Evaluate $\int\limits_{0}^{1} \dfrac{dx}{1+x}$ and correct to 3 decimal places using Romberg's method and hence find the value of $\log_e 2$.   (ND14)

4.13 Two point Gaussian quadrature formula

Two point Gaussian quadrature formula : (Guass two point formula)

Given $I = \int\limits_{a}^{b} f(x) dx$

Case (i) If $a = -1, b = +1$, then

$$I = \int\limits_{-1}^{1} f(x) dx = f\left[-\frac{1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right]$$

Case (ii) If $a = 0, b = 1$, then

$$I = \int_0^1 f(x)\,dx = \frac{1}{2}\int_{-1}^1 f(x)\,dx, \text{ if } f(x) \text{ is an even function}$$

Case (iii) If $(a \neq -1 \, \& \, b \neq 1)$, then $x = \dfrac{b-a}{2}z + \dfrac{b+a}{2} = mz + c \Rightarrow dx = m\,dz$

$$I = \int_a^b f(x)dx = \int_{-1}^1 f(z)\,m\,dz = m\int_{-1}^1 f(z)\,dz$$

$$= m\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$

4.13.1 Part A

1. Use two-point Gaussian quadrature formula to solve $\int_{-1}^1 \dfrac{dx}{1+x^2}$ & also find error. (MJ2010,AM2012)

   Solution : Given interval is −1 to 1 so we apply formula

   $$\int_{-1}^1 f(x)\,dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

   Here $f(x) = \dfrac{1}{1+x^2}$

   $$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

   $$f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

   $$\therefore \int_{-1}^1 \frac{1}{1+x^2}dx = \frac{3}{4} + \frac{3}{4} = \frac{3}{2} = 1.5$$

   But actual integration

   $$\int_{-1}^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_{-1}^1 = \tan^{-1}(1) - \tan^{-1}(-1)$$

   $$= \tan^{-1}(1) + \tan^{-1}(1)$$

   $$= 2\tan^{-1}(1)$$

   $$= 2\frac{\pi}{4} = \frac{\pi}{2}$$

   $$= 1.5708$$

   Here the error due to two-point formula is 0.0708.

2. Evaluate $\int_0^2 e^{-x^2}dx$ by two point Gaussian quadrature formula. (ND2010)

   Solution : Here $f(x) = e^{-x^2}$ [Given range is not in exact form]

   Let $x = \dfrac{b-a}{2}z + \dfrac{b+a}{2}$ [Here $a = 0, b = 2$]

   $$= \frac{2-0}{2}z + \frac{2+0}{2}$$

   $x = z + 1$ when $x = 0 \Rightarrow z = -1$

   $dx = dz$ when $x = 2 \Rightarrow z = 1$

$$\int_0^2 e^{-x^2} dx = \int_{-1}^1 e^{-(z+1)^2} dz$$

$$= f\left[\frac{-1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right] \tag{1}$$

Here $f(z) = e^{-(z+1)^2}$

$$f\left[\frac{-1}{\sqrt{3}}\right] = e^{-\left[\frac{-1}{\sqrt{3}}+1\right]^2} = e^{-0.1786} = 0.8364$$

$$f\left[\frac{1}{\sqrt{3}}\right] = e^{-\left[\frac{1}{\sqrt{3}}+1\right]^2} = e^{-2.488} = 0.0831$$

$$\therefore (1) \Rightarrow \int_0^2 e^{-x^2} dx = 0.8364 + 0.0831$$

$$= 0.9195$$

3. Write down two point Gaussian quadrature formula. (ND2011)

   Solution: $$\left[\int_{-1}^1 f(x) dx = f\left[-\frac{1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right]\right]$$

4. Evaluate $\int_{-2}^2 e^{\frac{-x}{2}}$ by Gauss two point formula. (ND2013)

   Solution: [4.6854]

4.13.2 Part B

Example 4.35. Evaluate $\int_1^2 \frac{dx}{x}$ by using Gaussian two point formula.

Solution: Here $a \neq -1, b = 2 \neq 1$,

$$\text{so use } x = \frac{(b-a)z + (b+a)}{2}$$

$$x = (z+3)/2 \Rightarrow dx = dz/2$$

$\therefore$ The above integral becomes

$$I = \int_1^2 \frac{dx}{x} = \int_{-1}^1 \frac{dz/2}{(z+3)/2} = \int_{-1}^1 \frac{1}{z+3} dz = \int_{-1}^1 f(z) \, dz, \text{ where } f(z) = \frac{1}{z+3}$$

$\therefore$ By Gaussian two point formula

$$\int_{-1}^1 f(z) \, dz = f\left(z = \frac{1}{\sqrt{3}}\right) + f\left(z = -\frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{1}{z+3}\right)_{z=\frac{1}{\sqrt{3}}} + \left(\frac{1}{z+3}\right)_{z=-\frac{1}{\sqrt{3}}}$$

$$= 0.693$$

Example 4.36. Using Gaussian two point formula evaluate $\int\limits_{0}^{\pi/2} \log(1 + x)dx$

Solution: $I = \int\limits_{-1}^{1} \log\left[1 + \frac{\pi}{4}(1 + z)\right]\frac{\pi}{4}dz$

$$= \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$= 0.858$$

Example 4.37. Evaluate $\int\limits_{1}^{2} \frac{dx}{1 + x^3}$ by using Gaussian two point formula.

Solution: $I = 0.2544.$

4.14 Three point Gaussian quadrature formula

Case (i) $\int\limits_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$

Case (ii) $\int\limits_{0}^{1} f(x)dx = \frac{1}{2}\int\limits_{-1}^{1} f(x)dx$ [for even function $f(x)$]

$$= -\frac{1}{2}\left\{\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)\right\}$$

Case (iii) $(a \neq -1 \& b \neq 1)$, then $x = \frac{b-a}{2}z + \frac{b+a}{2} = mz + c \Rightarrow dx = m\,dz$

$$I = \int\limits_{a}^{b} f(x)dx = \int\limits_{-1}^{1} f(z)\,m\,dz = m\left\{\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)\right\}$$

4.14.1 Part A

1. Write down the three point Gaussian quadrature formula to evaluate $\int\limits_{-1}^{1} f(x)dx.$ (ND2012)

     Solution: $\left[\text{Ans}: \int\limits_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)\right]$

4.14.2 Part B

Example 4.38. Evaluate $\int\limits_{1}^{2} \frac{1}{1 + x^3}dx$ by using Gaussian three point formula.

Solution: Using the substitution $x = \frac{(b-a)z + (b+a)}{2}, a = 1, b = 2$

$$x = \frac{z + 3}{2}$$

$$I = \int_{-1}^{1} \frac{1}{1 + \left(\frac{z+3}{2}\right)^3} \left(\frac{dz}{2}\right)$$

$$= \frac{1}{2} \int_{-1}^{1} \frac{1}{1 + \left(\frac{z+3}{2}\right)^3} dz$$

$$= \frac{8}{2} \int_{-1}^{1} \frac{1}{8 + (z+3)^3} dz$$

$$= \int_{-1}^{1} \frac{4}{8 + (z+3)^3} dz, \qquad \text{where } f(z) = \frac{4}{8 + (z+3)^3}$$

$$= \left\{ \frac{5}{9} \left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right) \right] + \frac{8}{9} f(0) \right\}$$

$$= \frac{5}{9} [0.27505] + \frac{8}{9} (0.11429)$$

$$= 0.25439 \qquad \left[\because \frac{5}{9} = 0.5555, \frac{8}{9} = 0.8888 \right]$$

$$I = 0.02544$$

Example 4.39. **Evaluate** $\int_{0}^{1} \frac{dx}{1 + x^2}$ **by two and three point Gaussian quadrature formula & hence find the value of π.**

Solution:

$$\text{Now}, I = \int_{0}^{1} \frac{dx}{1 + x^2} = \int_{-1}^{1} \frac{1}{2} \frac{1}{1 + x^2}$$

$$= \frac{1}{2} \int_{-1}^{1} f(x) dx, \qquad \text{where } f(x) = \frac{1}{2} \frac{1}{1 + x^2}$$

By Gaussian two point formula

$$I = \frac{1}{2} \int_{-1}^{1} f(x) dx = \frac{1}{2} \left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) \right] = 0.75 \qquad (1)$$

By Guassain three point formula

$$I = \frac{1}{2} \int_{-1}^{1} f(x) dx = \frac{1}{2} \left[\frac{5}{9} \left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right) \right] + \frac{8}{9} f(0) \right] = 0.79166 \qquad (2)$$

By actual integration

$$I = \int_{0}^{1} \frac{dx}{1 + x^2} = \left[\tan^{-1}(x) \right]_{0}^{1}$$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} \qquad (3)$$

From (2) & (3),

$$0.79166 = \frac{\pi}{4}$$

$$\Rightarrow \pi = 3.16664$$

Example 4.40. Find $\int\limits_{0}^{\frac{\pi}{2}} \sin x \, dx$ by two & three point Gaussian quadrature formula.

Solution: $x = \dfrac{\pi(z + 1)}{4}$

$I = 0.9985$ (by two point formula)

$I = 1.0000$ (by three point formula)

Example 4.41. Find $\int\limits_{0}^{1} \dfrac{1}{t} dt$ by using Gaussian three point formula.

Solution: $[I = 1.6027]$.

4.14.3 Anna University Questions

1. Evaluate $\int\limits_{1}^{2} \dfrac{1}{1 + x^3} dx$ using Using Gauss three point formula.   (AM11)
   Solution: [0.2544]

2. Evaluate $\int\limits_{0}^{2} \dfrac{x^2 + 2x + 1}{1 + (x + 1)^2} dx$ by Gaussian three point formula.   (MJ13)
   Solution: [1.5363]

3. Apply three point Gaussian quadrature formula to evaluate $\int\limits_{0}^{1} \dfrac{\sin x}{x} dx$.   (ND13)
   Solution: [0.94616]

4. Evaluate $\int\limits_{1}^{2} \dfrac{dx}{1 + x^2}$ using 3 point Gaussian formula.   (ND14)

4.15 Double integrals by Trapezoidal

Double integration by trapezoidal rule is

Given $\int\limits_{c}^{d} \int\limits_{a}^{b} f(x) \, dx \, dy [= I(Say)]$

$$\text{i.e., } I = \frac{hk}{4} \left\{ \begin{array}{l} [\text{sum of values of } f \text{ at the four corners}] \\ +2 \left[\begin{array}{l} \text{sum of values of } f \text{ at the nodes} \\ \text{on the boundary except the corners} \end{array} \right] \\ +4 \, [\text{sum of the values at the interior nodes}] \end{array} \right\}$$
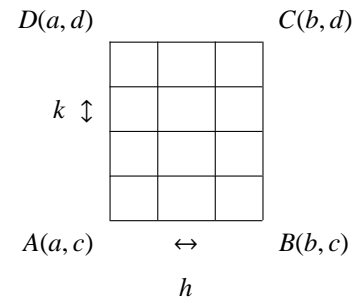
where $h = \dfrac{b-a}{n}, \; k = \dfrac{d-c}{m}$

where $n = $ is number of equal intervals in (a, b).

where $m = $ is number of equal intervals in (c, d).

4.15.1 Part B

Example 4.42.   Evaluate the integral $\int\limits_{1}^{1.4} \int\limits_{2}^{2.4} \dfrac{dx dy}{xy}$ using Trapezoidal rule. Verify your results by actual integration.

Solution: $f(x, y) = \dfrac{1}{xy}$, x varies from $(2, 2.4)$

y varies from $(1, 1.4)$

Divide the range of x and y into 4 equal parts.

$$h = \frac{2.4 - 2}{4} = 0.1, \; k = \frac{1.4 - 1}{4} = 0.1$$

The values of $f(x, y)$ at the nodal points are given in the table :

y \ x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3698	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

By Trapezoidal rule for double integration

$$I = \frac{hk}{4} \left[\begin{array}{l} \text{Sum of values of } f \text{ at the four corners} \\ +2 \left(\begin{array}{l} \text{Sum of values of } f \text{ at the nodes} \\ \text{on the boundary except the corners} \end{array} \right) \\ +4 \, (\text{Sum of the values at the interior nodes}) \end{array} \right]$$

$$= \frac{(0.1)(0.1)}{4} \left[\begin{array}{l} (0.5 + 0.4167 + 0.2976 + 0.3571) \\ +2 \left(\begin{array}{l} 0.4762 + 0.4545 + 0.4348 + 0.3788 + 0.3472 + 0.3205 \\ +0.3106 + 0.3247 + 0.3401 + 0.3846 + 0.4167 + 0.4545 \end{array} \right) \\ +4 \left(\begin{array}{l} 0.4329 + 0.4132 + 0.3953 + 0.3623 + 0.3344 \\ +0.3497 + 0.3663 + 0.3698 + 0.3788 \end{array} \right) \end{array} \right]$$

$$= 0.0614$$

By actual integration

$$\int_{1}^{1.4}\int_{2}^{2.4} \frac{1}{xy}dxdy = \int_{1}^{1.4}\left(\int_{2}^{2.4} \frac{1}{x}dx\right)\frac{1}{y}dy = \int_{1}^{1.4}(\log x)_{2}^{2.4}\,\frac{1}{y}dy$$

$$= (\log 2.4 - \log 2)(\log y)_{1}^{1.4}$$

$$= 0.0613$$

4.15.2 Anna University Questions

1. Evaluate $\int_{1}^{1.2}\int_{1}^{1.4} \frac{dxdy}{x+y}$ by trapezoidal formula by taking $h = k = 0.1$.   (AM10)
   Solution: [0.0349]

2. Evaluate $\int_{1}^{5}\left[\int_{1}^{4} \frac{1}{x+y}dx\right]dy$ by Trapezoidal rule in x–direction with $h = 1$ and Simpson's one-third
   rule in y–direction with $k = 1$.   (ND10)
   Solution: [By Trap. : $I = 2.4053$, Simp. : $I = 2.122$]

3. Evaluate $\int_{0}^{1}\int_{0}^{1} \frac{1}{x+y+1}dx\,dy$ by using Trapezoidal rule taking $h = 0.5$ and $k = 0.25$.   (AM11)
   Solution: [$0.5319 \approx 0.532$]

4. Using Trapezoidal rule, evaluate $\int_{1}^{2}\int_{1}^{2} \frac{dxdy}{x^2 + y^2}$ numerically with $h = 0.2$ along x–direction and $k =$
   0.25 along y–direction.   (MJ12)
   Solution: [0.2643]

5. Evaluate $\int_{2}^{2.4}\int_{4}^{4.4} xy\,dx\,dy$ by Trapezoidal rule taking $h = k = 0.1$.   (ND13)
   Solution: [1.4784]

6. Evaluate $\int_{0}^{1}\int_{0}^{1} \frac{1}{1+x+y}dxdy$.   (ND14)

4.16 Double integrals by Simpson's 1/3 rules

Double integration by Simpson's 1/3 rule is Given $\int_{c}^{d}\int_{a}^{b} f(x)\,dx\,dy[= I(S\,ay)]$

$$i.e., \ I = \frac{hk}{9} \left\{ \begin{array}{l} [\text{sum of values of } f \text{ at the four corners}] \\ +2 \left[\begin{array}{l} \text{sum of values of } f \text{ at the odd positions} \\ \text{on the boundary except the corners} \end{array} \right] \\ +4 \left[\begin{array}{l} \text{sum of the values of } f \text{ at the even positions} \\ \text{on the boudary except the corners} \end{array} \right] \\ +4 \left[\begin{array}{l} \text{sum of the values of } f \text{ at odd positions} \\ \text{on the odd rows of the matrix except boundary rows} \end{array} \right] \\ +8 \left[\begin{array}{l} \text{sum of the values of } f \text{ at even positions} \\ \text{on the odd rows of the matrix except boundary rows} \end{array} \right] \\ +8 \left[\begin{array}{l} \text{sum of the values of } f \text{ at odd positions} \\ \text{on the even rows of the matrix except boundary rows} \end{array} \right] \\ +16 \left[\begin{array}{l} \text{sum of the values of } f \text{ at even positions} \\ \text{on the even rows of the matrix except boundary rows} \end{array} \right] \end{array} \right\}$$

4.16.1 Part B

Example 4.43. Evaluate the integral $\int_{1}^{1.4} \int_{2}^{2.4} \frac{dxdy}{xy}$ using Simpson's rule. Verify your results by actual integration.

Solution: $f(x, y) = \frac{1}{xy}$, x varies from $(1, 1.4)$

y varies from $(2, 2.4)$

Divide the range of x and y into 4 equal parts.

$$h = \frac{2.4 - 2}{4} = 0.1, \ k = \frac{1.4 - 1}{4} = 0.1$$

The values of $f(x, y)$ at the nodal points are given in the table :

y \ x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3698	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

By Extended Simpson's rule

$$
I = \frac{hk}{9}
\begin{bmatrix}
\text{(Sum of the values of } f \text{ at the four corners)} \\
+ 2 \text{(Sum of the values of } f \text{ at the odd positions on the boundary except the corners)} \\
+ 4 \text{(Sum of the values of } f \text{ at the even positions on the boundary except the corners)} \\
+ 4 \left(\begin{array}{l} \text{Sum of the values of } f \text{ at the odd positions} \\ \text{on the odd rows of the matrix except boundary rows} \end{array} \right) \\
+ 8 \left(\begin{array}{l} \text{Sum of the values of } f \text{ at the even positions} \\ \text{on the odd rows of the matrix except boundary rows} \end{array} \right) \\
+ 8 \left(\begin{array}{l} \text{Sum of the values of } f \text{ at the odd positions} \\ \text{on the even rows of the matrix except boundary rows} \end{array} \right) \\
+ 16 \left(\begin{array}{l} \text{Sum of the values of } f \text{ at the even positions} \\ \text{on the even rows of the matrix except boundary rows} \end{array} \right)
\end{bmatrix}
$$

$$
= \frac{(0.1)(0.1)}{9}
\begin{bmatrix}
(0.5 + 0.4167 + 0.2976 + 0.3571) \\
+2(0.4545 + 0.3472 + 0.3247 + 0.4167) \\
+4 \left(\begin{array}{c} 0.4762 + 0.4348 + 0.3788 + 0.3205 + 0.3106 \\ +0.3401 + 0.3846 + 0.4545 \end{array} \right) \\
+4(0.3788) \\
+8(0.3968 + 0.3623) \\
+8(0.3497 + 0.4132) \\
+16(0.3663 + 0.3344 + 0.4329 + 0.3953)
\end{bmatrix}
$$

$$
= \frac{0.01}{9}(55.2116) = 0.0613
$$

By actual integration

$$
\int_1^{1.4} \int_2^{2.4} \frac{1}{xy} dx dy = \int_1^{1.4} \left(\int_2^{2.4} \frac{1}{x} dx \right) \frac{1}{y} dy = \int_1^{1.4} (\log x)_2^{2.4} \frac{1}{y} dy
$$

$$
= (\log 2.4 - \log 2)(\log y)_1^{1.4}
$$

$$
= 0.0613
$$

There is no error(or deviation) netween actual integration and Simpson's rule.

Example 4.44. **Evaluate** $I = \int_0^{0.5} \int_0^{0.5} \frac{\sin xy}{1 + xy} dx dy$ **by using Simpson's rule with step size 0.25.**

Solution: [Ans : $I = 0.000216$(Simpson's rule)].

4.16.2 Double integrals by Trapezoidal, Simpson rule

Example 4.45. **Evaluate the integral** $\int_1^{1.4} \int_2^{2.4} \frac{dx dy}{xy}$ **using Trapezoidal rule and Simpson's rule. Verify your results by actual integration.**

Solution: [$I = 0.0614$(Trapezoidal rule), $I = 0.0613$(Simpson's rule)]

Example 4.46. **Evaluate** $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x + y) dx dy$ **by using trapezoidal rule, Simpson's rule and also by actual integration.**

Solution: Divide into 2 intervals (3 ordinates) on x&y

By Trapezoidal rule, $I = 1.7975$

Simpson's rule, $I = 2.0080$

Actual integration, $I = 2$

4.16.3 Anna University Questions

1. Evaluate $\int\limits_{1}^{5}\left[\int\limits_{1}^{4}\dfrac{1}{x+y}dx\right]dy$ by Trapezoidal rule in x–direction with $h = 1$ and Simpson's one-third rule in y–direction with $k = 1$.   (ND10)

   Solution: [By Trap. : $I = 2.4053$, Simp. : $I = 2.122$]

2. Evaluate $\int\limits_{0}^{2}\int\limits_{0}^{1}4xy\,dx\,dy$ using Simpson's rule by taking $h = \dfrac{1}{4}$ and $k = \dfrac{1}{2}$.   (ND12)

   Solution: [3.1111]

3. Evaluate $\int\limits_{1}^{1.4}\int\limits_{2}^{2.4}\dfrac{1}{xy}dx\,dy$ using Simpson's one-third rule.   (MJ13)

   Solution: [0.0613]

4. Taking $h = k = \dfrac{1}{4}$, evaluate $\int\limits_{0}^{\frac{1}{2}}\int\limits_{0}^{\frac{1}{2}}\dfrac{\sin(xy)}{1+xy}dx\,dy$ using Simpson's rule .   (AM14)

   Solution: [0.0141]

4.16.4 Assignment problems

1. Given the following data, find $y'(6)$ and the maximum value of y (if it exists).   (AM10)

x :	0	2	3	4	7	9
y :	4	26	58	112	466	922

2. Find $f'(x)$ at $x = 1.5$ and $x = 4.0$ from the following data using Newton's formulae for differentiation.

x :	1.5	2.0	2.5	3.0	3.5	4.0
$y = f(x)$:	3.375	7.0	13.625	24.0	38.875	59.0

   (MJ12)

3. Find the first three derivatives of $f(x)$ at $x = 1.5$ by using Newton's forward interpolation formula to the data given below.

x :	1.5	2	2.5	3	3.5	4
y :	3.375	7	13.625	24	38.875	59

   (MJ13)

4. Using Trapezoidal rule, evaluate $\int\limits_{-1}^{1}\dfrac{1}{(1+x^2)}dx$ by taking eight equal intervals.   (MJ13)

5. The velocity v of a particle at a distance S from a point on its path is given by the table below:

S (meter)	0	10	20	30	40	50	60
v (m / sec)	47	58	64	65	61	52	38

   (AM10)

Estimate the time taken to travel 60 meters by Simpson's $1/3^{rd}$ rule and Simpson's $3/8^{th}$ rule.

6. Evaluate $I = \int\limits_{0}^{6} \dfrac{1}{1+x}dx$ by using (i) direct integration (ii) Trapezoidal rule (iii) Simpson's one-third rule (iv) Simpson's three-eighth rule. (ND11)

7. Taking $h = 0.05$ evaluate $\int\limits_{1}^{1.3} \sqrt{x}dx$ using Trapezoidal rule and Simpson's three-eighth rule. (AM14)

8. Use Romberg's method to compute $\int\limits_{0}^{1} \dfrac{1}{1+x^2}dx$ correct to 4 decimal places. Also evaluate the same integral using tree-point Gaussian quadrature formula. Comment on the obtained values by comparing with the exact value of the integral which is equal to $\dfrac{\pi}{4}$. (MJ12)

9. Evaluate $\int\limits_{0}^{\frac{1}{2}} \dfrac{x}{\sin x}dx$ correct to three decimal places using Romberg's method. (AM14)

10. Evaluate $\int\limits_{0}^{1} \dfrac{dx}{1+x}$ and correct to 3 decimal places using Romberg's method and hence find the value of $\log_e 2$. (ND14)

11. Evaluate $\int\limits_{0}^{2} \dfrac{x^2 + 2x + 1}{1 + (x+1)^2}dx$ by Gaussian three point formula. (MJ13)

12. Apply three point Gaussian quadrature formula to evaluate $\int\limits_{0}^{1} \dfrac{\sin x}{x}dx$. (ND13)

13. Evaluate $\int\limits_{0}^{1}\int\limits_{0}^{1} \dfrac{1}{x+y+1}dx\,dy$ by using Trapezoidal rule taking $h = 0.5$ and $k = 0.25$. (AM11)

14. Using Trapezoidal rule, evaluate $\int\limits_{1}^{2}\int\limits_{1}^{2} \dfrac{dxdy}{x^2+y^2}$ numerically with $h = 0.2$ along x–direction and $k = 0.25$ along y–direction. (MJ12)

15. Evaluate $\int\limits_{2}^{2.4}\int\limits_{4}^{4.4} xy\,dx\,dy$ by Trapezoidal rule taking $h = k = 0.1$. (ND13)

16. Taking $h = k = \dfrac{1}{4}$, evaluate $\int\limits_{0}^{\frac{1}{2}}\int\limits_{0}^{\frac{1}{2}} \dfrac{\sin(xy)}{1+xy}dxdy$ using Simpson's rule . (AM14)

5 Numerical Solution of Ordinary Differential Equations

Single Step methods – Taylor's series method – Euler's method – Modified Euler's method

– Fourth order Runge – Kutta method for solving first order equations

Multi step methods – Milne's and Adams – Bash forth predictor corrector methods

for solving first order equations

5.1 Introduction

Single Step methods	Multi step methods
Taylor's series method	Milne's forth predictor corrector method
Taylor's series method for simultaneous first ODE	Adams-Bash forth predictor corrector method
Taylor's series method for II order ODE	
Euler's method	
Modified Euler's method	
Runge-Kutta method for first order ODE	
Runge - Kutta method for solving II order ODE	
R. K. Method for simultaneous first order ODE	

5.2 Taylor's series method

$$\text{Given } \frac{dy}{dx} = y' = f(x, y) \text{ with } y(x_0) = y_0$$

$$\Rightarrow \frac{d^2 y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\text{i.e., } y'' = f_x + f_y(y')$$

$\text{lll}^{ly} \text{ find } y''', y'''', \cdots$

Taylor's series expansion of $y(x)$ above $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \cdots$$

$$= y_0 + \frac{(x-x_0)}{1!}y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \cdots$$

(or)

$$y(x_1) = y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots, \quad \text{where } x_1 = x_0 + h, h = x_1 - x_0$$

$$y(x_2) = y_2 = y_1 + \frac{h}{1!}y'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \cdots, \quad \text{where } x_2 = x_1 + h, h = x_2 - x_1$$

5.2.1 Part A

1. Using Taylor series method find $y(1.1)$ given that $y' = x + y, y(1) = 0$.   (MJ2011)

   Solution: Given $y' = x + y, y(x = 1) = 0 \Rightarrow x_0 = 1, y_0 = 0, x_1 = 1.1, h = x_1 - x_0 = 1.1 - 1 = 0.1$

   ∴ Taylor's series formula is

   $$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y'^v_0 + \cdots \tag{1}$$

$y' = x + y$	$y'_0 = x_0 + y_{0(x_0=1,y_0=0)} = 1 + 0 = 1$	(2)
$y'' = 1 + y'$	$y''_0 = 1 + y'_0 = 1 + 1 = 2$	[by (2)] (3)
$y''' = 0 + y'' = y''$	$y'''_0 = y''_0 = 2$	[by (3)] (4)
$y'^v = y'''$	$y'^v_0 = y'''_0 = 2$	[by (4)]

   $$\therefore (1) \Rightarrow y_1 = 0 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2)$$

   $$\text{i.e., } y(1.1) = (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12} + \cdots$$

   $$= 0.1103083$$

   $$\equiv 0.11031$$

2. What is the major drawback of Taylor series method?   (AM2012)

   Solution: The major drawbacks of Taylor series method are :

   1. In the differential equation, $\frac{dy}{dx} = f(x, y)$ may have a complicated algebraic structure.

   2. The evaluation of higher order derivatives may become tedious.

3. Find $y(0.1)$ if $\frac{dy}{dx} = 1 + y, y(0) = 1$ using Taylor series method.   (ND2012)

   Solution: Given $y' = 1 + y, y(x = 0) = 1 \Rightarrow x_0 = 0, y_0 = 1, x_1 = 0.1, h = x_1 - x_0 = 0.1 - 0 = 0.1$

   ∴ Taylor's series formula is

   $$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y'^v_0 + \cdots \tag{1}$$

$y' = 1 + y$	$y'_0 = 1 + y_{0(x_0=0,y_0=1)} = 1 + 1 = 2$	(2)
$y'' = 0 + y' = y'$	$y''_0 = y'_0 = 2$	[by (2)] (3)
$y''' = y''$	$y'''_0 = y''_0 = 2$	[by (3)] (4)
$y'^v = y'''$	$y'^v_0 = y'''_0 = 2$	[by (4)]

$$\therefore (1) \Rightarrow y_1 = 1 + \frac{0.1}{1!}(2) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2)$$

$$\text{i.e., } y(1.1) = 1.210341667$$

$$\equiv 1.21034$$

4. Find $y(1.1)$ if $y' = x + y, y(1) = 0$ by Taylor series method.   (AM2013)

   Solution: $[Ref : (MJ2011)]$

5. State the advantages and disadvantages of the Taylor's series method.   (AM2014)

   Solution:

   The advantages of the Taylor's series method are :

   1. It is a powerful single step method if we are able to find the successiove derivatives easily.

   2. This method gives a straight forward adaptation of classic calculus to develop the solution as an infinite aseries.

   3. This method will be very useful for finding the starting values for powerful methods like Runge-Kutta method, Milne's method etc.

   The disadvantages of the Taylor's series method are :

   1. In the differential equation, $\frac{dy}{dx} = f(x, y)$ may have a complicated algebraic structure.

   2. The evaluation of higher order derivatives may become tedious.

5.2.2 Part B

Example 5.1. Using Taylor's series method, find y at $x = 0.1$, if $\frac{dy}{dx} = x^2y - 1, y(0) = 1$

Solution: Given $y' = x^2y - 1$, $x_0 = 0$, $y_0 = 1, h = 0.1$ (G)

$\therefore$ Taylor's series formula is

$$y_1 = y_0 + \frac{h}{1!}y' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y'^v_0 + \cdots \tag{1}$$

$y' = x^2y - 1$	$y'_0 = x_0^2y_0 - 1\big|_{(x_0=0,y_0=1)} = 0 - 1 = -1$ [by (G)] (2)
$y'' = 2xy + x^2y'$	$y''_0 = 2x_0y_0 + x_0^2y'_0 = 0$ [by (G) & (2)] (3)
$y''' = 2(xy' + y) + x^2y'' + y'2x$	$y'''_0 = 2(x_0y'_0 + y_0) + x_0^2y''_0 + y'_02x_0$ $= 2$ [by (G) & (3)] (4)
$y^{iv} = \begin{array}{l} 2y' + 2xy'' + 2y' + 2xy'' \\ + x^2y'' + y''(2x) + 2y' \end{array}$	$y^{iv}_0 = \begin{array}{l} 2y'_0 + 2x_0y''_0 + 2y'_0 + 2x_0y''_0 \\ + x_0^2y''_0 + y''_0(2x_0) + 2y'_0 \end{array}$ [by (G) & (4)] $= -6$

$$\therefore (1) \Rightarrow y_1 = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(-6)$$

$$\text{i.e., } y(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

$$= 0.900308$$

Example 5.2. Use Taylor series method to solve $\dfrac{dy}{dx} = x - y^2$, $y(0) = 1$ and hence compute $y(0.1)$ and $y(0.2)$ correct to 4 places of decimals.

Solution: Given $y' = x - y^2$ and $x_0 = 0, y_0 = 1$.

Take $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y(x_1) = y(0.1)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0'^{v} + \cdots \tag{1}$$

$y' = x - y^2$	$\begin{aligned} y_0' &= x_0 - y_0^2 \\ &= 0 - 1 = -1 \end{aligned}$
$y'' = 1 - 2yy'$	$\begin{aligned} y_0'' &= 1 - 2y_0y_0' \\ &= 1 - 2(1)(-1) = 3 \end{aligned}$
$\begin{aligned} y''' &= -2(yy'' + y'y') \\ &= -2yy'' - 2y'^2 \end{aligned}$	$\begin{aligned} y_0''' &= -2y_0y_0'' - 2y_0'^2 \\ &= -2(1)(3) - 2 = -8 \end{aligned}$
$\begin{aligned} y^{iv} &= -2(yy''' + y''y') - 4y'y'' \\ &= -6y'y'' - 2yy''' \end{aligned}$	$\begin{aligned} y_0^{iv} &= -6y_0'y_0'' - 2y_0y_0''' \\ &= -6(-1)(3) - 2(1)(-8) = 34 \end{aligned}$

From (1)

$$\therefore (1) \Rightarrow y_1 = 1 + \frac{0.1}{1}(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8) + \frac{(0.1)^4}{24}(34)$$

$$= 1 - 0.1 + 0.015 - 0.001333 + 0.0001417$$

$$= 0.9138$$

$$\therefore y(0.1) = 0.9138$$

To find y_2

$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$

$$y_2 = y(x_2) = y(0.2)$$

The Taylor's series formula for y_2 is

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1'^{v} + \cdots \tag{2}$$

$y' = x - y^2$	$\begin{aligned} y_1' &= x_1 - y_1^2 \\ &= 0.1 - (0.9138)^2 = -0.735 \end{aligned}$
$y'' = 1 - 2yy'$	$\begin{aligned} y_1'' &= 1 - 2y_1y_1' \\ &= 1 - 2(0.9138)(-0.735) = 2.3433 \end{aligned}$
$\begin{aligned} y''' &= -2(yy'' + y'y') \\ &= -2yy'' - 2y'^2 \end{aligned}$	$\begin{aligned} y_1''' &= -2y_1y_1'' - 2y_1'^2 \\ &= -2(0.9138)(2.3433) - 2(-0.735)^2 \\ &= -5.363 \end{aligned}$

$y^{iv} = -2(yy''' + y''y') - 4y'y''$ $\quad = -6y'y'' - 2yy'''$	$y_0^{iv} = -6y_1'y_1'' - 2y_1y_1'''$ $\quad = -6(-0.735)(2.3433) - 2(0.9138)(-5.363)$ $\quad = 20.1354$

From (2)

$$y_2 = 0.9138 + \frac{0.1}{1}(-0.735) + \frac{(0.1)^2}{2}(2.3433) + \frac{(0.1)^3}{6}(-5.363) + \frac{(0.1)^4}{24}(20.1354)$$

$$y_2 = 0.8512$$

$$\therefore y(0.2) = 0.8512$$

Example 5.3. **Using Taylor series method, find y to five places of decimals when $x = 1.3$ given that $dy = \left(x^2y - 1\right)dx$ and $y = 2$ when $x = 1$.**

Solution: Given $y' = x^2y - 1$ and $x_0 = 1, y_0 = 2$.

Given $h = 0.3$

$$x_1 = x_0 + h = 1 + 0.3 = 1.3$$

$$y_1 = y(x_1) = y(1.3)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots \tag{1}$$

$y' = x^2y - 1$	$y_0' = x_0^2y_0 - 1$ $\quad = 1^2(2) - 1 = 1$
$y'' = x^2y' + 2xy$	$y_0'' = x_0^2y_0' + 2x_0y_0$ $\quad = 1^2(1) + 2(1)(2) = 5$
$y''' = x^2y'' + 2xy' + 2[xy' + y]$ $\quad = x^2y'' + 4xy' + 2y$	$y_0''' = x_0^2y_0'' + 4x_0y_0' + 2y_0$ $\quad = 1^2(5) + 4(1)(1) + 2(2) = 13$
$y_0^{iv} = x^2y''' + 2xy'' + 4[xy'' + y'] + 2y'$ $\quad = x^2y''' + 6xy'' + 6y'$	$y_0^{iv} = x_0^2y_0''' + 6x_0y_0'' + 6y_0'$ $\quad = 1^2(13) + 6(1)(5) + 6(1) = 49$

From (1),

$$y_1 = 2 + \frac{0.3}{1!}(1) + \frac{(0.3)^2}{2!}(5) + \frac{(0.3)^3}{3!}(13) + \frac{(0.3)^4}{4!}(49)$$

$$= 2 + 0.3 + 0.225 + 0.0585 + 0.01654$$

$$= 2.60004$$

Example 5.4. **Using Taylor series method find y at $x = 0.1$ correct to four decimal places from $\frac{dy}{dx} = x^2 - y, y(0) = 1$ with $h = 0.1$. Compute terms upto x^4.**

Solution: Given $y' = x^2 - y$ and $x_0 = 0, y_0 = 1$.

Given $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y(x_1) = y(0.1)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0'^v + \cdots \tag{1}$$

$y' = x^2 - y$	$y_0' = x_0^2 - y_0$ $= 0^2 - 1 = -1$
$y'' = 2x - y'$	$y_0'' = 2x_0 - y_0'$ $= 2(0) - (-1) = 1$
$y''' = 2 - y''$	$y_0''' = 2 - y_0''$ $= 2 - 1 = 1$
$y'^v = -y'''$	$y_0'^v = -y_0'''$ $= -1$

From (1),

$$y_1 = 1 + \frac{0.1}{1!}(-1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(-1)$$

$$= 1 - 0.1 + 0.005 + 0.000167 - 0.000004$$

$$= 0.905163$$

Example 5.5. Using Taylor series method, find $y(1.1)$ and $y(1.2)$ correct to four decimal places given $\dfrac{dy}{dx} = xy^{\frac{1}{3}}$ and $y(1) = 1$.

Solution: Given $y' = xy^{\frac{1}{3}}$ and $y(1) = 1$.

Given $h = 0.1$

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

$$y_1 = y(x_1) = y(1.1)$$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots \tag{1}$$

$y' = xy^{\frac{1}{3}}$	$y_0' = x_0 y_0^{\frac{1}{3}}$ $= 1(1)^{\frac{1}{3}} = 1$

$y'' = x\dfrac{1}{3}y^{-\frac{2}{3}}y' + y^{\frac{1}{3}}$ $= \dfrac{1}{3}x^2 y^{-\frac{1}{3}} + y^{\frac{1}{3}}$	$y''_0 = \dfrac{1}{3}x_0^2 y_0^{-\frac{1}{3}} + y_0^{\frac{1}{3}}$ $= \dfrac{1}{3}(1)^2(1)^{-\frac{1}{3}} + (1)^{\frac{1}{3}} = \dfrac{4}{3}$
$y''' = \dfrac{1}{3}\left[x^2\left(-\dfrac{1}{3}\right)y^{-\frac{4}{3}}y' + y^{-\frac{1}{3}}(2x)\right]$ $+ \dfrac{1}{3}y^{-\frac{2}{3}}y'$ $= -\dfrac{1}{9}x^2 y^{-\frac{4}{3}}y' + \dfrac{2}{3}xy^{-\frac{1}{3}} + \dfrac{1}{3}y^{-\frac{2}{3}}y'$	$y'''_0 = -\dfrac{1}{9}x_0^2 y_0^{-\frac{2}{3}}y'_0 + \dfrac{2}{3}x_0 y_0^{-\frac{1}{3}}$ $+ \dfrac{1}{3}y_0^{-\frac{2}{3}}y'_0$ $= -\dfrac{1}{9}(1)^2(1)^{-\frac{2}{3}}(1)$ $+ \dfrac{2}{3}(1)(1)^{-\frac{1}{3}} + \dfrac{1}{3}(1)^{-\frac{2}{3}}(1)$ $= \dfrac{8}{9}$

From (1),

$$y_1 = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}\left(\frac{4}{3}\right) + \frac{(0.1)^3}{3!}\left(\frac{8}{9}\right)$$

$$= 1 + 0.1 + 0.00666 + 0.000148$$

$$= 1.10681$$

$$\therefore y(1.1) = 1.10681$$

To find y_2

$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

$y_1 = y(x_2) = y(1.2)$

The Taylor's series formula for y_2 is

$$y_2 = y_1 + \frac{h}{1!}y'_1 + \frac{h^2}{2!}y''_1 + \frac{h^3}{3!}y'''_1 + \cdots \qquad (2)$$

$y' = xy^{\frac{1}{3}}$	$y'_1 = x_1 y_1^{\frac{1}{3}}$ $= (1.1)(1.10681)^{\frac{1}{3}} = 1.13785$
$y'' = \dfrac{1}{3}x^2 y^{-\frac{1}{3}} + y^{\frac{1}{3}}$	$y''_1 = \dfrac{1}{3}x_1^2 y_1^{-\frac{1}{3}} + y_1^{\frac{1}{3}}$ $= \dfrac{1}{3}(1.1)^2(1.10681)^{-\frac{1}{3}} + (1.10681)^{\frac{1}{3}}$ $= 1.42433$

$y''' = -\dfrac{1}{9}x^2 y^{-\frac{4}{3}} y' + \dfrac{2}{3}xy^{-\frac{1}{3}}$ $+\dfrac{1}{3}y^{-\frac{2}{3}}y'$	$y'''_1 = -\dfrac{1}{9}x_1^2 y_1^{-\frac{4}{3}} y'_1 + \dfrac{2}{3}x_1 y_1^{-\frac{1}{3}} + \dfrac{1}{3}y_1^{-\frac{2}{3}} y'_1$ $= -\dfrac{1}{9}(1.1)^2(1.10681)^{-\frac{4}{3}}(1.13785)$ $+\dfrac{2}{3}(1.1)(1.10681)^{-\frac{1}{3}}$ $+\dfrac{1}{3}(1.10681)^{-\frac{2}{3}}(1.13785)$ $= 0.92979$

From (2),

$$y_2 = 1.10681 + \frac{0.1}{1!}(1.13785) + \frac{(0.1)^2}{2!}(1.42433) + \frac{(0.1)^3}{3!}(1.92979)$$

$$= 1.2277$$

$$\therefore y(1.2) = 1.2277$$

Example 5.6. **Find the Taylor series solution with three terms for the initial value problem** $\dfrac{dy}{dx} = x^3 + y, y(1) = 1.$

Solution: Given $y' = x^3 + y$ and $x_0 = 1, y_0 = 1$.

The Taylor's series formula for y_1 is

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}y'(x_0) + \frac{(x - x_0)^2}{2!}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \tag{1}$$

$y' = x^3 + y$	$y'_0 = x_0^3 + y_0$ $= 1^3 + 1 = 2$
$y'' = 3x^2 + y'$	$y''_0 = 3x_0^2 + y'_0$ $= 3(1^2) + 2 = 5$
$y''' = 6x + y''$	$y'''_0 = 6x_0 + y''_0$ $= 6(1) + 5 = 11$

Substituting in (1)

$$y_1 = 1 + \frac{x-1}{1!}(2) + \frac{(x-1)^2}{2!}(5) + \frac{(x-1)^3}{3!}(11)$$

$$= 1 + 2(x-1) + \frac{5}{4}(x-1)^2 + \frac{11}{6}(x-1)^3$$

Example 5.7. **Using Taylor's method, compute** $y(0.2)$ **and** $y(0.4)$ **correct to 4 decimal places given** $\dfrac{dy}{dx} (= y') = 1 - 2xy$ **&** $y(0) = 0.$

Solution: $y(0.2) = 0.194752003, y(0.4) = 0.359883723$

Example 5.8. **Find the Taylor series solution of** $y(0.1)$ **given that** $\dfrac{dy}{dx} + y^2 = e^x, y(0) = 1.$ **Compute using first five terms.**

Solution: Given $y' = e^x - y^2$ and $x_0 = 0, y_0 = 1$.

Take $h = 0.1$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y(x_1) = y(0.1)$

To find y_1

The Taylor's series formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \frac{h^4}{4!}y^{iv}_0 + \frac{h^5}{5!}y^v_0 + \cdots \tag{1}$$

$y' = e^x - y^2$	$\begin{aligned} y'_0 &= e^{x_0} - y_0^2 \\ &= e^0 - 1^2 = 0 \end{aligned}$
$y'' = e^x - 2yy'$	$\begin{aligned} y''_0 &= e^{x_0} - 2y_0y'_0 \\ &= e^0 - 2(1)(0) = 1 \end{aligned}$
$\begin{aligned} y''' &= e^x - 2[yy'' + y'y'] \\ &= e^x - 2[yy'' + y'^2] \end{aligned}$	$\begin{aligned} y'''_0 &= e^{x_0} - 2[y_0y''_0 - y'^2_0] \\ &= e^0 - 2[1(1) + 0^2] = -1 \end{aligned}$
$\begin{aligned} y^{iv} &= e^x - 2[yy''' + y''y' + 2y'y''] \\ &= e^x - 2[y'y''' + 3y'y''] \end{aligned}$	$\begin{aligned} y^{iv}_0 &= e^{x_0} - 2[y'_0y'''_0 + 3y'_0y''_0] \\ &= e^0 - 2[1(-1) + 3(0)(1)] = 3 \end{aligned}$
$\begin{aligned} y^v &= e^x - 2[yy^{iv} + y'''y' \\ &\quad + 3(y'y''' + y''y'')] \\ &= e^x - 2[y'y^{iv} + 4y'y''' + 3y''^2] \end{aligned}$	$\begin{aligned} y^v_0 &= e^{x_0} - 2[y'_0y^{iv}_0 + 4y'_0y'''_0 + 3y''^2_0] \\ &= 1 - 2[1(3) + 4(0)(-1) + 3(1)^2] \\ &= -11 \end{aligned}$

From (1)

$$y_1 = 1 + \frac{0.1}{1}(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(-1) + \frac{(0.1)^4}{24}(3) + \frac{(0.1)^5}{120}(-11)$$

$$= 1 + 0 + 0.005 - 0.000167 + 0.000125 - 0.000000917$$

$$= 1.00496$$

$$\therefore y(0.1) = 1.00496$$

Example 5.9. **Using Taylor's series method, with the first five terms in the expansion; find $y(0.1)$ correct to 3 decimal places, given that $\dfrac{dy}{dx} = e^x - y^2, y(0) = 1$**

Solution: $y(0.1) = 1.0049891 \cong 1.005$ (correct to 3 decimal places)

5.2.3 Anna University Questions

1. Evaluate the value of y at $x = 0.1, 0.2$ given $\dfrac{dy}{dx} = x^2y - 1, y(0) = 1$, by Taylor's series method upto four terms. (ND10)

   Solution: $[y(0.1) = 0.900308, y(0.2) = 0.802269]$

2. Using Taylor series method to find $y(0.1)$ if $y' = x^2 + y^2, y(0) = 1$. (MJ13)

   Solution: $[y(0.1) = 1.1115]$

3. Obtain y by Taylor series method, given that $y' = xy + 1, y(0) = 1$, for $x = 0.1$ and 0.2 correct to four decimal places.   (ND13)

   Solution: $[y(0.1) = 1.1053, y(0.2) = 1.2224]$

4. Using Taylor's series method, find y at $x = 1.1$ by solving the equation $\dfrac{dy}{dx} = x^2 + y^2; y(1) = 2$. Carryout the computations upto fourth order derivative.   (AM14)

   Solution: $[y(0.1) = 2.64333]$

5. Using Taylor's series method, find y at $x = 0$ if $\dfrac{dy}{dx} = x^2 y - 1, y(0) = 1$.   (ND14)

5.3 Taylor's series method for simultaneous first order differential equations

Given $\dfrac{dy}{dx} = f_1(x, y, z), \dfrac{dz}{dx} = f_2(x, y, z)$ with initial conditions $y(x_0) = y_0, z(x_0) = z_0$

5.3.1 Part B

Example 5.10. **Solve the system of equations $\dfrac{dy}{dx} = z - x^2, \dfrac{dz}{dx} = y + x$ with $y(0) = 1, z(0) = 1$ by taking $h = 0.1$ to get $y(0.1)$ and $z(0.1)$.**

Solution: Given $x_0 = 0, y_0 = 1, z_0 = 1$

WKT, Taylor's series for y_1 is

$$y_1 = y(0.1) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \cdots \tag{1}$$

& Taylor's series for z_1 is

$$z_1 = z(0.1) = z_0 + \frac{h}{1!}z_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \frac{h^4}{4!}z_0^{iv} + \cdots \tag{2}$$

$y' = \dfrac{dy}{dx} = z - x^2$	$y_0' = z_0 - x_0^2 = 1 - 0 = 1$	$z' = \dfrac{dz}{dx} = x + y$	$z_0' = x_0 + y_0 = 0 + 1 = 1$
$y'' = z' - 2x$	$y_0'' = 1$	$z'' = 1 + y'$	$z_0'' = 1 + y_0' = 2$
$y''' = z'' - 2$	$y_0''' = 0$	$z''' = y''$	$z_0''' = 1$
$y^{iv} = z'''$	$y_0^{iv} = 1$	$z^{iv} = y'''$	$z_0^{iv} = 0$

$$\therefore (1) \Rightarrow y(0.1) = 1 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(1)$$

$$= 1.1050 \qquad\qquad\qquad\qquad\qquad \text{[correct to 4 decimal places]}$$

$$\text{Now } (2) \Rightarrow z(0.1) = 1 + \frac{(0.1)}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(1) + \frac{(0.1)^4}{4!}(0)$$

$$= 1.110167$$

$$\simeq 1.1102 \qquad\qquad\qquad\qquad\qquad \text{[correct to 4 decimal places]}$$

5.4 Taylor's series for II order differential equations

5.4.1 Part B

Example 5.11. By Taylor's series, find $y(0.1)$ and $y(0.2)$ given that $y'' = y + xy', y(0) = 1, y'(0) = 0$.

Solution: Given $x_0 = 0, y_0 = 1, y'_0 = 0$ & $y'' = y + xy'$

WKT, Taylor's series formula is

$$y(x) = y_0 + xy'_0 + \frac{x^2}{2!}y''_0 + \frac{x^3}{3!}y'''_0 + \frac{x^4}{4!}y^{iv}_0 + \cdots \tag{1}$$

$y'' = y + xy'$	$y''_0 = y_0 + x_0y'_0 = 1 + 0(0) = 1$
$y''' = y' + xy'' + y' = 2y' + xy''$	$y'''_0 = 2(0) + (0)(1) = 0$
$y^{iv} = 2y'' + xy''' + y'' = 3y'' + xy'''$	$y^{iv}_0 = 3(1) + (0)(0) = 3$

$$\therefore (1) \Rightarrow y(x) = 1 + 0 + \frac{x^2}{2!}(1) + 0 + \frac{x^4}{4!}(3) + \cdots$$

$$= 1 + \frac{x^2}{2} + \frac{x^4}{8} + \cdots$$

$$y(0.1) = 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{8} + \cdots$$

$$= 1.0050125 \simeq 1.0050 \qquad \text{(correct to 4 decimal places)}$$

$$y(0.2) = 1 + \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} + \cdots$$

$$= 1.0202$$

5.5 Euler's method

Given $y' = f(x, y), x_0, y_0, h$

Euler algorithm is

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$\vdots$$

In general, $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

5.5.1 Part A

1. Use Euler's method to find $y(0.2)$ and $y(0.4)$ given $\frac{dy}{dx} = x + y, y(0) = 1$. (MJ2010)

    Solution : Given $f(x, y) = x + y, x_0 = 0, y_0 = 1, x_1 = 0.2, x_2 = 0.4$.   (Here $h = x_1 - x_0 = x_2 - x_1 = 0.2$)

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.2)[x_0 + y_0] = 1 + (0.2)[0 + 1]$$

i.e., $y_1 = y(0.2) = 1.2$

$$y_2 = y_1 + hf(x_1, y_1) = 1.2 + (0.2)[x_1 + y_1] = 1.2 + (0.2)[0.2 + 1.2]$$

$$= 1.2 + 0.28$$

i.e., $y_2 = y(0.4) = 1.48$

2. Find $y(0.1)$ by using Euler's method given that $\dfrac{dy}{dx} = x + y, y(0) = 1$.   (ND10)

   Solution : Given, $\dfrac{dy}{dx} = x + y = f(x, y), x_0 = 0, y_0 = 1, x_1 = 0.1, h = 0.1$

   By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.1)[x_0 + y_0]$$

$$= 1 + (0.1)[0 + 1]$$

$$= 1 + 0.1$$

$$= 1.01$$

   i.e., $y(0.1) = 1.01$

3. Find $y(0.2)$ for the equation $y' = y + e^x$, given that $y(0) = 0$ by using Euler's method.   (AM11)

   Solution : Given, $f(x, y) = y + e^x, x_0 = 0, y_0 = 0, h = 0.2$

   By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 0 + 0.2f(0, 0)$$

$$= 0.2\left[0 + e^0\right] = 0.2$$

   i.e., $y(0.2) = 0.2$

4. State Euler's method to solve $\dfrac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$.   (ND11)

   Solution: $[y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})]$

5. Using Euler's method, find the solution of the initial value problem $\dfrac{dy}{dx} = \log(x + y), y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$.   (MJ12)

   Solution : Given, $\dfrac{dy}{dx} = \log(x + y) = f(x, y), x_0 = 0, y_0 = 2, h = 0.2$

   By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 2 + 0.2f(0, 2)$$

$$= 2 + 0.2[\log(0 + 2)] = 2 + 0.2[\log 2] = 2 + 0.2(0.30103)$$

   i.e., $y(0.2) = 2.060206$

6. State Euler's formula. (MJ13)

Solution: $[y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})]$

7. Using Euler's method find the solution of the initial value problem $y' = y - x^2 + 1, y(0) = 0.5$ at $x = 0.2$ taking $h = 0.2$. (ND13)

Solution : Given, $y' = y - x^2 + 1 = f(x, y), x_0 = 0, y_0 = 0.5, h = 0.2$

By Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 0.5 + 0.2f(0, 0.5)$$
$$= 0.5 + 0.2\left[y_0 - x_0^2 + 1\right] = 0.5 + 0.2\left[0.5 - 0^2 + 1\right] = 2 + 0.2(1.5)$$

i.e., $y(0.2) = 0.8$

8. Given $y' = x + y, y(0) = 1$, find $y(0.1)$ by Euler's method. (ND14)

5.5.2 Part B

Example 5.12. Using Euler's method, find $y(0.2)$, $y(0.4)$ and $y(0.6)$ from $\dfrac{dy}{dx} = x + y, y(0) = 1$ with $h = 0.2$.

Solution: Given $y(0)\,[(\text{or})\,y_0] = 1 \Rightarrow x_0 = 0$

$$\& x_1 = 0.2, x_2 = 0.4, x_3 = 0.4, h = 0.2$$
$$y' = x + y$$

We have to find $y(0.2) = y_1$

Now, by Euler algorithm,

$$y_1 = y_0 + hf(x_0, y_0)$$
$$\text{i.e., } y(0.2) = [1 + (0.2)(x + y)]_{\left(\substack{x=x_0 \\ y=y_0}\right)} = 1 + 0.2(x_0 + y_0) = 1 + (0.2)(0 + 1)$$
$$y(0.2) = 1.2 = y_1$$
$$y_2 = y_1 + hf(x_1, y_1) = 1.2 + (0.2)\left[x_1 + y_1\right]$$
$$= 1.2 + 0.2\,[0.2 + 1.2] = 1.48 \Rightarrow y(0.4) = 1.48$$
$$y_3 = y_2 + hf(x_2, y_2) = 1.48 + (0.2)\left[x_2 + y_2\right]$$
$$= 1.856 \Rightarrow y(0.6) = 1.856$$

$\therefore$ The result of the problem:

x	y
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.2$	$y_1 = y_0 + hf(x_0, y_0) = 1.2$
$x_2 = 0.4$	$y_2 = y_1 + hf(x_1, y_1) = 1.48$
$x_3 = 0.4$	$y_3 = y_2 + hf(x_2, y_2) = 1.856$

Example 5.13.   Using Euler's method solve $y' = x + y + xy, y(0) = 1$ **compute** y **at** $x = 0.1$ **by taking** $h = 0.05.$

Solution: Given $y' = x + y + xy, x_0 = 0, y_0 = 1$

$h = 0.05.$

$\qquad x_1 = x_0 + h = 0 + 0.05 = 0.05$

$\qquad y_1 = y(x_1) = y(0.05)$

To find y_1

   By Euler algorithm

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1 + 0.05 f(0, 1)$$
$$= 1 + 0.05[0 + 1 + (0)(1)]$$
$$y_1 = 1.05$$
$$\therefore y(0.05) = 1.05$$

$\qquad x_2 = x_1 + h = 0.05 + 0.05 = 0.1$

$\qquad y_2 = y(x_2) = y(0.1)$

To find y_2

   By Euler algorithm

$$y_2 = y_1 + hf(x_1, y_1)$$
$$= 1.05 + 0.05 f(0.05, 1.05)$$
$$= 105 + 0.05[0.05 + 1.05 + (0.05)(1.05)]$$
$$y_2 = 1.10762$$
$$\therefore y(0.1) = 1.10762$$

Example 5.14.   Using Euler's method find $y(0.3)$ **of** $y(x)$ **satisfies the initial value problem** $\dfrac{dy}{dx} = \dfrac{1}{2}\left(x^2 + 1\right)y^2, y(0.2) = 1.1114.$

Solution: Given $f(x, y) = \dfrac{1}{2}\left(x^2 + 1\right)y^2$ and $x_0 = 0.2, y_0 = 1.1114.$

Take $h = 0.1$

$\qquad x_1 = x_0 + h = 0.2 + 0.1 = 0.3$

$\qquad y_1 = y(x_1) = y(0.3)$

To find y_1

By Euler's method

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 1.1114 + 0.1 f(0.2, 1.1114)$$

$$= 1.1114 + 0.1 \left[\frac{1}{2} \left((0.2)^2 + 1 \right) (1.1114)^2 \right]$$

$$y_1 = 1.1756$$

$$\therefore y(0.3) = 1.1756$$

Example 5.15. **Using Euler's method with $h = 0.1$ to solve the equation $\dfrac{dy}{dx} = \dfrac{y}{1+x}, y(0) = 2$ in the range $0 \leq x \leq 1$.**

Solution: Given $f(x, y) = \dfrac{y}{1+x}$ and $x_0 = 0, y_0 = 2, h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = y(x_1) = y(0.1)$$

To find y_1

By Euler algorithm

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 2 + 0.1 f(0, 2)$$

$$= 2 + 0.1 \left[\frac{2}{1+0} \right]$$

$$y_1 = 2.2$$

$$\therefore y(0.1) = 2.2$$

Similarly

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$	$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$	$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$
$y_2 = y(x_2) = y(0.2)$	$y_3 = y(x_3) = y(0.3)$	$y_4 = y(x_4) = y(0.4)$
$\therefore y_2 = y_1 + hf(x_1, y_1)$	$\therefore y_3 = y_2 + hf(x_2, y_2)$	$\therefore y_4 = y_3 + hf(x_3, y_3)$
$= 2.2 + 0.1 f(0.1, 2.2)$	$= 2.4 + 0.1 f(0.2, 2.4)$	$= 2.6 + 0.1 f(0.3, 2.6)$
$= 2.2 + 0.1 \left[\dfrac{2.2}{1+0.1} \right]$	$= 2.4 + 0.1 \left[\dfrac{2.4}{1+0.2} \right]$	$= 2.6 + 0.1 \left[\dfrac{2.4}{1+0.3} \right]$
$y_2 = 2.4$	$y_3 = 2.6$	$y_4 = 2.8$
$\therefore y(0.2) = 2.4$	$\therefore y(0.3) = 2.6$	$\therefore y(0.3) = 2.8$
$y_5 = y_4 + hf(x_4, y_4)$	$y_6 = y_5 + hf(x_5, y_5)$	$y_7 = y_6 + hf(x_6, y_6)$
$= 2.8 + 0.1 f(0.4, 2.8)$	$= 3 + 0.1 f(0.5, 3)$	$= 3.2 + 0.1 f(0.6, 3.2)$
$= 2.8 + 0.1 \left[\dfrac{2.8}{1+0.4} \right]$	$= 3 + 0.1 \left[\dfrac{3}{1+0.5} \right]$	$= 3.2 + 0.1 \left[\dfrac{3.2}{1+0.6} \right]$
$y_5 = 3$	$y_6 = 3.2$	$y_7 = 3.4$
$\therefore y(0.5) = 3$	$\therefore y(0.6) = 3.2$	$\therefore y(0.7) = 3.4$

$y_8 = y_7 + hf(x_7, y_7)$	$y_9 = y_8 + hf(x_8, y_8)$	$y_{10} = y_9 + hf(x_9, y_9)$
$= 3.4 + 0.1 f(0.7, 3.4)$	$= 3.6 + 0.1 f(0.8, 3.6)$	$= 3.8 + 0.1 f(0.9, 3.8)$
$= 3.4 + 0.1 \left[\dfrac{3.4}{1 + 0.7}\right]$	$= 3.6 + 0.1 \left[\dfrac{3.6}{1 + 0.8}\right]$	$= 3.8 + 0.1 \left[\dfrac{3.8}{1 + 0.9}\right]$
$y_8 = 3.6$	$y_9 = 3.8$	$y_{10} = 4$
$\therefore y(0.8) = 3.6$	$\therefore y(0.9) = 3.8$	$\therefore y(1) = 4$

Example 5.16. Solve $\dfrac{dy}{dx} = y + e^x, y(0) = 0$ for $x = 0.2, 0.4$ by using Euler's method.

Solution: $[y_1 = 0.2, y_2 = 0.484281]$

Example 5.17. Using Euler's method, find the solution of the initial value problem $\dfrac{dy}{dx} =$ $\log(x + y), y(0) = 2$ at $x = 0.2$ by assuming $h = 0.2$.

Solution: $[y(0.2) = 2.0602]$

5.5.3 Anna University Questions

1. Solve $y' = \dfrac{y - x}{y + x}, y(0) = 1$ at $x = 0.1$ by taking $h = 0.02$ by using Euler's method.   (MJ13)

   Solution: $[y(0.02) = 1.02, y(0.04) = 1.0392, y(0.06) = 1.0577, y(0.08) = 1.0756, \therefore y(0.1) = 1.0928]$

5.6 Modified Euler's method

Modified Euler's method formula is

$$\text{In general, } y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]$$

$$\text{When } n = 0, y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$\text{When } n = 1, y_2 = y_1 + hf\left[x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x, y)\right]$$

$$\text{When } n = 2, y_3 = y_2 + hf\left[x_2 + \frac{h}{2}, y_2 + \frac{h}{2} f(x_2, y_2)\right]$$

5.6.1 Part B

Example 5.18. Solve $y' = 1 - y, y(0) = 0$ by modified Euler method.

Solution: Given $y' = f(x, y) = 1 - y, x_0 = 0, y_0 = 0$.

Let $h = 0.1, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$

We have to find y_1, y_2, y_3.

By Modified Euler's method:

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2}, y_n + \frac{1}{2}hf(x_n, y_n)\right]$$

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right] \qquad [\text{Here } f(x_0, y_0) = f(x = x_0, y = y_0) = 1 - y_0 = 1 - 0 = 1]$$

$$\therefore y_1 = 0 + (0.1) f\left[0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}(1)\right]$$

$$= 0.1 f [0.05, 0.05]$$

$$= \{(0.1)[1 - y]\}_{\left(\substack{x=0.05 \\ y=0.05}\right)} = 1 - 0.05$$

$$y_1 = 0.095$$

Now $y_2 = y_1 + hf\left[x_1 + \dfrac{h}{2}, y_1 + \dfrac{h}{2}f(x_1, y_1)\right]$ \qquad $[\text{Here } f(x_1, y_1) = 1 - y_1 = 0.905]$

i.e., $y_2 = 0.905 + (0.1) f\left[0.1 + \dfrac{0.1}{2}, 0.095 + \dfrac{0.1}{2}f(0.1, 0.095)\right]$

$$y_2 = 0.18098$$

Now $y_3 = y_2 + hf\left[x_2 + \dfrac{h}{2}, y_2 + \dfrac{h}{2}f(x_2, y_2)\right]$ \quad $[\text{Here } f(x_2, y_2) = 1 - y_2 = 1 - 0.18098 = 0.81902]$

i.e., $y_3 = 0.18098 + (0.1) f\left[0.2 + \dfrac{0.1}{2}, 0.18098 + \dfrac{0.1}{2}f(0.2, 0.18098)\right]$

$$y_3 = 0.258787$$

$$\boxed{\therefore y_0 = 0, y_1 = 0.095, y_2 = 0.18098, y_3 = 0.258787}$$

Example 5.19. By using modified Euler method, find $y(1.8)$ if $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}, y(2) = 1$ [**Hint** $h = -0.2$]

Solution: $\hspace{8cm} [y(1.8) = 0.9349].$

Example 5.20. Given $\dfrac{dy}{dx} + y - x^2 = 0, y(0) = 1, y(0.1) = 0.9052, y(0.2) = 0.8213$, **find correct to four decimal places** $y(0.3)$, **using modified Euler's method.**

Solution: $\left[\text{Hint } y' = x^2 - y\right]$ $\qquad \left[\text{Ans}: y(0.3) = y_3 = y_2 + hf\left[x_2 + \dfrac{h}{2}, y_2 + \dfrac{h}{2}f(x_2, y_2)\right] = 0.7493\right]$

5.6.2 Anna University Questions

1. Using Modified Euler's method, find $y(4.1)$ and $y(4.2)$ if $5x\dfrac{dy}{dx} + y^2 - 2 = 0; y(4) = 1$. \hfill (ND12)

   Solution: $\hspace{6cm} [y(4.1) = 1.005, \ y(4.2) = 1.0098]$

2. Apply modified Euler's method to find $y(0.2)$ and $y(0.4)$ given $y' = x^2 + y^2, y(0) = 1$ by taking $h = 0.2$. \hfill (ND14)

5.7 Fourth order Runge-Kutta method for solving first order equations

Fourth order Runge-Kutta method for solving I order Differential Equations
[Single step method]

$$\text{Given } \frac{dy}{dx} = y' = f(x, y)$$

$$\& \ y(x_0) = y_0$$

We have to find $y(x_1) = ? = y_1, y(x_2) = ? = y_2, y(x_3) = ? = y_3, \cdots$

To find $f(x_{n+1}) = y(x_{n+1}) = y_{n+1}$:

$$y_{n+1} = y(x_n + h) = y(x_n) + \Delta y = y_n + \Delta y$$

$$\text{where } \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$\text{where } k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

To find $f(x_1) = y(x_1) = y_1$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$\therefore y_1 = y(x_0 + h) = y(x_0) + \Delta y = y_0 + \Delta y$$

5.7.1 Part A

1. State the fourth order Runge - Kutta algorithm.   (ND2012)

   Solution: The fourth order Runge - Kutta algorithm is

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$\therefore y_1 = y(x_0 + h) = y(x_0) + \Delta y = y_0 + \Delta y$$

5.7.2 Part B

Example 5.21. Given $\dfrac{dy}{dx} = x + y^2, y(0) = 1$**, find** $y(0.1)$ **&** $y(0.2)$ **by Runge-Kutta method for IV order.**

Solution: Given $y' = f(x, y) = x + y^2, h = 0.1$

$$\& x_0 = 0, y(x_0 = 0) = 1$$

To find $y(0.1) = y_1$

$$k_1 = hf[x_0, y_0] = (0.1)f(0.1) = (0.1)\left[0 + 1^2\right] = 0.1$$

$$k_2 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right] = (0.1)f[0 + 0.05, (1 + 0.05)] = (0.1)\left[0.05 + (1.05)^2\right] = 0.11525$$

$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right] = (0.1)f\left[0.05, 1 + \frac{0.11525}{2}\right] = (0.1)\left[0.05 + (1.057625)^2\right] = 0.116857$$

$$k_4 = hf[x_0 + h, y_0 + k_3] = (0.1)f[0.05, 1 + 0.116857] = (0.1)\left[0.05 + (1.116857)^2\right] = 0.134737$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = \frac{1}{6}[(0.1) + 2(0.1) + 2(0.11525) + (0.116857)]$$

$$= 0.11649$$

$\therefore y(0.1) = y_1 = y_0 + \Delta y = 1 + 0.11649 = 1.11649 \Rightarrow y_1 = 1.11649$

To find $y(0.2) = y_2$

$$k_1 = hf[x_1, y_1] = (0.1)f[0.1, 1.11649] = (0.1)[0.1 + (1.116492)] = 0.1347$$

$$k_2 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right] = (0.1)f\left[0.1 + 0.05, 1.11649 + \frac{0.1347}{2}\right] = (0.1)f[0.15, 1.18385] = 0.1552$$

$$k_3 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right] = (0.1)f\left[0.1 + 0.05, 1.11649 + \frac{0.1552}{2}\right] = (0.1)f[0.15, 1.1941] = 0.1576$$

$$k_4 = hf[x_1 + h, y_1 + k_3] = (0.1)f[0.1 + 0.1, 1.11649 + 0.1576] = 0.18233$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = 0.1571$$

$$y_2 = y_1 + \Delta y = 1.11649 + 0.1571 = 1.27359$$

i.e.,

x	0	0.1	0.2
y	1	1.11649	1.27359

Example 5.22. Using R.K. method of fourth order solve for $x = 1.4$**, from** $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ **with** $y = 0$ **at** $x = 1.$

Solution: $[y_1 = y(1.2) = 0.140, \therefore y_2 = y(1.4) = 0.27, \text{ with } h = 0.2]$

Example 5.23. Solve y **for** $x = 0.2, 0.4$ **from the given O.D.E.** $\dfrac{dy}{dx} = \sqrt{x^2 + y}$ **and** $y = 0.8$ **at** $x = 0.$
Solution: $[y_1 = y(0.2) = 0.99029, \ y_2 = y(0.4) = 0.120828]$

5.7.3 Anna University Questions

1. Using Runge-Kutta method of order four, find y when $x = 1.2$ in steps of 0.1 given that $y' = x^2 + y^2$
   and $y(1) = 0.5$.   (ND13)

   Solution: $[y_1 = y(1.1) = y_0 + \nabla y = 0.6428, \therefore y_2 = y(1.2) = y_1 + \nabla y = 0.8278$ with $h = 0.1]$

5.8 Fourth order Runge - Kutta method for solving II order differential equation

Given $f(y'', y', y) = g(x)$ with $y(x_0), y'(x_0)$. Find $y(x_1) = y_1, y'(x_1) = y_1'$ [or $z(x_1) = z_1$].

Let us set $y' = z = f_1(x, y, z)$

$\qquad y'' = f_2(x, y, z)$

To find y_1	To find z_1
$k_1 = hf_1(x_0, y_0, z_0)$ $k_2 = hf_1\left[x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{\ell_1}{2}\right]$ $k_3 = hf_1\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{\ell_2}{2}\right]$ $k_4 = hf_1[x_0 + h, y_0 + k_3, z_0 + \ell_3]$	$\ell_1 = hf_2(x_0, y_0, z_0)$ $\ell_2 = hf_2\left[x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{\ell_1}{2}\right]$ $\ell_3 = hf_2\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{\ell_2}{2}\right]$ $\ell_4 = hf_2[x_0 + h, y_0 + k_3, z_0 + \ell_3]$
$\Delta y = \dfrac{1}{6}[k_1 + 2k_1 + 2k_2 + 2k_3]$ $y_1 = y_0 + \Delta y$	$\Delta z = \dfrac{1}{6}[\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4]$ $z_1 = z_0 + \Delta z$

Example 5.24. Consider the II order initial value problem $y'' - 2y' + 2y = e^{2t} \sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using fourth order Runge-Kutta method, find $y(0.2), z(0.2)$.

Solution: Let $t = x$. $\therefore$ Given equations can be written as

$y'' - 2y' + 2y = e^{2x} \sin x$ (1)

$\qquad y(x = 0) = -0.4$

$\qquad y'(x = 0) = -0.6$

Here $h = 0.2$

Setting $y' = z, y'' = z'$

$$\therefore (1) \Rightarrow z' = 2z - 2y + e^{2x} \sin x$$

Let $f_1(x, y, z) = \dfrac{dy}{dx} = z$,

$\quad \& f_2(x, y, z) = \dfrac{d^2 y}{dx^2} = \dfrac{dz}{dx} = 2z - 2x + e^{2x} \sin x$

$k_1 = hf_1(x_0, y_0, z_0) = (0.2)f_1(0, -0.4, -0.6)$ $\quad = (0.2)(z_0)$ $\quad = (0.2)(-0.6) = -0.12$	$\ell_1 = hf_2(x_0, y_0, z_0) = (0.2)f_2(0, -0.4, -0.6)$ $\quad = (0.2)\left[2z_0 - 2y_0 + e^{2x_0}\sin x_0\right]$ $\quad = (0.2)\left[2(-0.6) - 2(-0.4) + e^{2(0)}\sin(0)\right]$ $\quad = -0.08$
$k_2 = hf_1\left[x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{\ell_1}{2}\right]$ $\quad = (0.2)f_1\left[0 + \dfrac{0.2}{2}, (-0.4) - \dfrac{0.12}{2}, (-0.6) - \dfrac{0.8}{2}\right]$ $\quad = (0.2)f_1[0.1, -0.46, -0.64]$ $\quad = (0.2)(-0.64)$ $\quad = -0.128$	$\ell_2 = hf_2\left[x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{\ell_1}{2}\right]$ $\quad = (0.2)f_2\left[0 + \dfrac{0.2}{2}, -0.4 - \dfrac{0.12}{2}, -0.6 - \dfrac{0.08}{2}\right]$ $\quad = (0.2)f_2[0.1, -0.46, -0.64]$ $\quad = (0.2)\left[2(-0.64) - 2(-0.46) + e^{2(0.1)}\sin(0.1)\right]$ $\quad = -0.0476$
$k_3 = hf_1\left[x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_2}{2}, z_0 + \dfrac{\ell_2}{2}\right]$ $\quad = (0.2)f_1\left[0 + \dfrac{0.2}{2}, -0.4 - \dfrac{0.128}{2}, -0.6 - \dfrac{0.0476}{2}\right]$ $\quad = (0.2)f_1[0.1, -0.464, -0.62381]$ $\quad = (0.2)[-0.62381]$ $\quad = -0.12476$	$\ell_3 = hf_2\left[x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_2}{2}, z_0 + \dfrac{\ell_2}{2}\right]$ $\quad = (0.2)f_2\left[0 + \dfrac{0.2}{2}, -0.4 - \dfrac{0.128}{2}, -0.6 - \dfrac{0.0476}{2}\right]$ $\quad = (0.2)f_2[0.1, -0.464, -0.62381]$ $\quad = (0.2)\left[2(-0.62381) - 2(-0.464) + e^{2(0.1)}\sin(0.1)\right]$ $\quad = -0.03954$
$k_4 = hf_1[x_0 + h, y_0 + k_3, z_0 + \ell_3]$ $\quad = (0.2)f_1[0 + 0.2, -0.4 - 0.12476, -0.6 - 0.03954]$ $\quad = (0.2)f_1[0.2, -0.52476, -0.63954]$ $\quad = (0.2)[-0.63954]$ $\quad = -0.12791$	$\ell_4 = hf_2[x_0 + h, y_0 + k_3, z_0 + \ell_3]$ $\quad = (0.2)f_2[0 + 0.2, -0.4 - 0.12476, -0.6 - 0.03954]$ $\quad = (0.2)f_2[0.2, -0.52476, -0.63954]$ $\quad = (0.2)\left[2(-0.63954) - 2(-0.52476) + e^{2(0.2)}\sin(0.2)\right]$ $\quad = 0.013366$
$\Delta y = \dfrac{1}{6}[k_1 + 2k_1 + 2k_2 + 2k_3]$ $\quad = -0.12557$ $y_1 = y(0.2) = y_0 + \Delta y$ $\quad = -0.52557$	$\Delta z = \dfrac{1}{6}[\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4]$ $\quad = -0.04015$ $z_1 = z(0.2) = z_0 + \Delta z$ $\quad = -0.64015$

Example 5.25. Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, find the value of $y(0.1)$ by using R. K. method of fourth order.

Solution: $\hspace{8cm} [y(0.1) = 0.9950].$

Example 5.26. Given $y'' - x(y')^2 + y^2 = 0$, $y(0) = 1$, $y'(0) = 0$, find the value of $y(0.2)$ by using R. K. method of fourth order.

Solution: $\hspace{8cm} [y(0.2) = 0.9801].$

5.8.1 Anna University Questions

1. Find the value of $y(0.1)$ by Runge-Kutta method of fourth order given $y'' + xy' + y = 0$, $y(0) = 1$ and $y'(0) = 0$. $\hspace{4cm}$ (ND10)
   Solution: $\hspace{4cm} [y_1 = y(0.1) = y_0 + \nabla y = 0.9950]$

2. Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$. Find the value of $y(0.1)$ by using Runge-Kutta method of fourth order. $\hspace{4cm}$ (ND11)
   Solution: $\hspace{4cm} [y_1 = y(0.1) = y_0 + \nabla y = 0.9950]$

3. Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t} \sin t$ with $y(0) = -0.4$ and
   $y'(0) = -0.6$ using Fourth order Runge Kutta algorithm, find $y(0.2)$.   (MJ12)

   Solution: $[y_1 = y(0.2) = y_0 + \nabla y = -0.5159]$

4. Using Runge-Kutta method find $y(0.2)$ if $y'' = xy'^2 - y^2, y(0) = 1, y'(0) = 0, h = 0.2$.   (MJ13)

   Solution: $[y_1 = y(0.2) = y_0 + \nabla y = 0.9801]$

5. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ find the value of $y(0.1)$ by Runge-Kutta's method of fourth
   order.   (ND14)

   Solution: $[y_1 = y(0.1) = y_0 + \nabla y = 0.9950]$

5.9 Fourth order R. K. Method for simultaneous first order differential equations

Solving the equation $\dfrac{dy}{dx} = f_1(x, y, z) \& \dfrac{dz}{dx} = f_2(x, y, z)$ with the initial conditions $y(x_0) = y_0, z(x_0) = z_0$.
Now starting from with increments, (x_0, y_0, z_0) with increments, $\Delta y \& \Delta z$ in $y \& z$ respectively.
Use formula

$k_1 = hf_1(x_0, y_0, z_0)$	$\ell_1 = hf_2(x_0, y_0, z_0)$
$k_2 = hf_1\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{\ell_1}{2}\right)$	$\ell_2 = hf_2\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{\ell_1}{2}\right)$
$k_3 = hf_1\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_2}{2}, z_0 + \dfrac{\ell_2}{2}\right)$	$\ell_3 = hf_2\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_2}{2}, z_0 + \dfrac{\ell_2}{2}\right)$
$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + \ell_3)$	$\ell_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + \ell_3)$
$\Delta y = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	$\Delta z = \dfrac{1}{6}(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$
$y_1 = y_0 + \Delta y$	$z_1 = z_0 + \Delta z$

5.9.1 Part B

Example 5.27. Solving the system of differential equation $\dfrac{dy}{dx} = xz + 1, \dfrac{dz}{dx} = -xy$ for $x = 0.3$ using fourth order R. K. method, the initial values are $x = 0, y = 0, z = 1$.

Solution: Given $x_0 = 0, y_0 = 0, z_0 = 1, h = 0.3$.

$f_1(x, y, z) = xz + 1$	$f_2(x, y, z) = -xy$
$k_1 = hf_1(x_0, y_0, z_0) = (0.3)(x_0 z_0 + 1)$	$\ell_1 = hf_2(x_0, y_0, z_0) = (0.3)[-0(0)]$
$\quad = (0.3)(0 + 1) = 0.3$	$\quad = 0$
$k_2 = hf_1\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{\ell_1}{2}\right)$	$\ell_2 = hf_2\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_1}{2}, z_0 + \dfrac{\ell_1}{2}\right)$
$\quad = 0.345$	$\quad = -0.007$
$k_3 = hf_1\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_2}{2}, z_0 + \dfrac{\ell_2}{2}\right)$	$\ell_3 = hf_2\left(x_0 + \dfrac{h}{2}, y_0 + \dfrac{k_2}{2}, z_0 + \dfrac{\ell_2}{2}\right)$
$\quad = 0.3448$	$\quad = -0.0078$
$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + \ell_3)$	$\ell_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + \ell_3)$
$\quad = 0.3893$	$\quad = -0.031032$
$\Delta y = \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.34482$	$\Delta z = \dfrac{1}{6}(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4) = -0.01011$
$y_1 = y_0 + \Delta y = 0 + 0.34482$	$z_1 = z_0 + \Delta z = 1 + (-0.01011)$
$\Rightarrow y(0.3) = 0.34482$	$\Rightarrow z(0.3) = 0.98989$

Example 5.28. Solve the simultaneous differential equation $\dfrac{dy}{dx} = 2y + z, \dfrac{dz}{dx} = y - 3z, y(0) = 0, z(0) = 0.5$ for $y(0.1)$ and $z(0.1)$ using R. K. method of fourth method.

Solution: Given $x_0 = 0, y_0 = 0, z_0 = 0.5, h = 0.1$

$$[y(0.1) = 0.04814, z(0.1) = 0.37263].$$

Example 5.29. Using the R. K. method, tabulate the solution of the system $\dfrac{dy}{dx} = x + z, \dfrac{dz}{dx} = x - y, y = 0, z = 0,$ when $x = 0$ at intervals of $h = 0.1$ from $x = 0.0$ to $x = 0.2$.

Solution: Given $x_0 = 0, y_0 = 0, z_0 = 1, h = 0.1$

$$[\text{Hint: } y(0.1) = 0.1050, z(0.1) = 0.9998$$
$$y(0.2) = 0.2199, z(0.2) = 0.9986]$$

5.9.2 Anna University Questions

1. Solve $y(0.1)$ and $z(0.1)$ from the simultaneous differential equations $\dfrac{dy}{dx} = 2y + z; \dfrac{dz}{dx} = y - 3z; y(0) = 0, z(0) = 0.5$ using Runge-Kutta method of the fourth order. (ND12)

   Solution: Given $x_0 = 0, y_0 = 0, z_0 = 0.5, h = 0.1$

   $$[y(0.1) = 0.04814, z(0.1) = 0.37263].$$

Multi step methods

$h(= \text{Interval length})$

Given $(x_0, y_0), x_1, x_2, x_3, x_4$:

y_1, y_2, y_3 are given (or) known by Singular method

i.e.,

$y_{4,p}$ = known by Multistep predictor method

$y_{4,c}$ = known by Multistep corrector method

Given x :	x_0	x_1	x_2	x_3	x_4	x_4
Given or find y :	y_0	y_1	y_2	y_3	$y_{4,p}$	$y_{4,c}$
					find by Multistep predictor method	find by Multistep corrector method

Multi step methods:

1. Milne's forth predictor corrector methods for solving first order equations

2. Adam's Bash-forth predictor corrector methods for solving first order equations

5.10 Milne's forth predictor corrector methods for solving first order equations

Milne's predictor and corrector methods (multistep method)

Milne's predictor formula

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

Milne's corrector formula

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

5.10.1 Part A

1. What are multi-step methods? How are they better than single step methods?   (ND2010)

   Solution : 1. Milne's predictor - correction method.

              2. Adams-Bashforth predictor - correction method.

   In the single step methods, it is not possible to get any information about truncation error.

   In the multi step methods, it is possible to get easily a good estimate of the truncation error.

2. State the Milne's predictor and corrector formulae.   (AM2014)

3. State the Milne's predictor-corrector formulae.   (ND14)

5.10.2 Part B

Example 5.30. **Given** $\dfrac{dy}{dx} = \dfrac{1}{2}[x + y]$, $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$ **by Milne's method to find** $y(2)$.

Solution: Given

$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$	$x_4 = 2$
$y_0 = 2$	$y_1 = 2.636$	$y_2 = 3.595$	$y_3 = 4.968$	$y_4 = ?$

Here $h = 0.5$ Given $y' = \dfrac{1}{2}[x + y] = f(x, y)$ By Milne's predictor formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

$$\text{when } n = 3, y_{3+1,p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] \tag{1}$$

Now, $y'_1 = \frac{1}{2}[x_1 + y_1] = \frac{1}{2}[0.5 + 2.636] = 1.568$

$y'_2 = \frac{1}{2}[x_2 + y_2] = \frac{1}{2}[1 + 3.595] = 2.2975$

$y'_3 = \frac{1}{2}[x_3 + y_3] = \frac{1}{2}[1.5 + 4.968] = 3.234$

$(1) \Rightarrow y_{4,p} = 2 + \frac{4(0.5)}{3}[2(1.568) - 2.2975 + 2(3.234)]$

$= 6.871 \, (= y_4, \text{say})$

By Milne's corrector formula,

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

$$\text{when } n = 3, y_{4,c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right] \tag{2}$$

Now, $y'_4 = \frac{1}{2}[x_4 + y_4] = \frac{1}{2}[2 + 6.871] = 4.4355$

$(2) \Rightarrow y_{4,c} = 3.595 + \frac{0.5}{3}[2.2975 + 4(3.234) + 4.4355]$

$= 3.595 + \frac{0.5}{3}(19.669) = 6.8732$

$\therefore$ Corrected value of y at 2 is $y(2) = 6.8732$.

Example 5.31. Given $\dfrac{dy}{dx} = x^3 + y, y(0) = 2$, the values of $y(0.2) = 2.073, y(0.4) = 2.452, y(0.6) = 3.023$ are got by R.K. Method. Find $y(0.8)$ by Milne's method.

Solution: $\left[y_{4,p} = 4.1659, y_{4,c} = 3.7953\right]$

Example 5.32. Solve $y' = x - y^2, 0 \le x \le 1, y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795, y(0.6) = 0.1762$ by Milne's method to find $y(0.8)$ and $y(1)$.

Solution: $[y(0.8) = 0.3046, y(1) = 0.4515].$

5.10.3 Anna University Questions

1. Use Milne's predictor-corrector formula to find $y(0.4)$, given $\dfrac{dy}{dx} = \dfrac{(1 + x^2)y^2}{2}, y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12$ and $y(0.3) = 1.21$. (AM10)

   Solution: $\left[y_{4,p} = 1.2771, y_{4,c} = 1.2797\right]$

2. Given that $\dfrac{dy}{dx} = \dfrac{1}{2}\left(1 + x^2\right)y^2; y(0) = 1; y(0.1) = 1.06; y(0.2) = 1.12$ and $y(0.3) = 1.21$, evaluate $y(0.4)$ and $y(0.5)$ by Milne's predictor corrector method. (ND11)

   Solution: $\left[y_{4,p} = 1.2771, y_{4,c} = 1.2797\right]$

3. Given that $\dfrac{dy}{dx} = 1 + y^2; y(0.6) = 0.6841, y(0.4) = 0.4228, y(0.2) = 0.2.27, y(0) = 0$, find $y(-0.2)$ using Milne's method. (ND12)

   Solution: $\left[y_{4,p} = -0.2003, y_{4,c} = -0.2027\right]$

4. Use Milne's method to find $y(0.8)$, given $y' = \dfrac{1}{x + y}, y(0) = 2, y(0.2) = 2.0933, y(0.4) = 2.1755, y(0.6) = 2.2493$. (ND13)

   Solution: $\left[y_{4,p} = 2.3162, y_{4,c} = 2.3164\right]$

5. Given $5xy' + y^2 = 2, y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097, y(4.3) = 1.0143$. Compute $y(4.4)$ using Milne's method.   (ND14)

   Solution: $\left[y_{4,p} = 1.01897, y_{4,c} = 1.01874 \right]$

5.11 Adams-Bash forth predictor corrector methods for solving first order equations

Adams predictor and corrector methods(efficient multistep method) [Adam's-Bashforth method]

Adam's predictor formula :

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3} \right]$$

Adam's corrector formula :

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2} \right]$$

5.11.1 Part A

1. Write the Adam-Bashforth predictor and corrector formulae.   (MJ2010)

   (Or) State Adam's predictor-corrector formulae.   (ND2011)

   Solution : Adam's predictor and corrector formulas are

   $$y_{k+1,p} = y_k + \frac{h}{24}\left[55y'_k - 59y'_{k-1} + 37y'_{k-2} - 9y'_{k-3} \right]$$
   $$y_{k+1,c} = y_k + \frac{h}{24}\left[9y'_{k+1} + 19y'_k - 5y'_{k-1} + y'_{k-2} \right]$$

5.11.2 Part B

Example 5.33. Given $\dfrac{dy}{dx} = x^2 (1 + y), y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979$, evaluate $y(1.4)$ by Adam's-Basforth method.

Solution: Given

$x_0 = 1$	$x_1 = 1.1$	$x_2 = 1.2$	$x_3 = 1.3$	$x_4 = 1.4$
$y_0 = 1$	$y_1 = 1.233$	$y_2 = 1.548$	$y_3 = 1.979$	$y_4 =?$

Here $h = 0.1$

By Adam's predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3} \right]$$
$$\text{when } n = 3, y_{4,p} = y_3 + \frac{h}{24}\left[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0 \right] \qquad (1)$$

Here $y'_0 = x_0^2 (1 + y_0) = 1^2 [1 + 1] = 2$

$y'_1 = x_1^2 (1 + y_1) = (1.1)^2 [1 + 1.233] = 2.70193$

$y'_2 = x_2^2 (1 + y_2) = (1.2)^2 [1 + 1.548] = 3.66912$

$y'_3 = x_3^2 (1 + y_3) = (1.3)^2 [1 + 1.979] = 5.0345$

$(1) \Rightarrow y_{4,p} = 1.979 + \dfrac{0.1}{24} \left[55 \,(5.0345) - 59 \,(3.66912) + 37 \,(2.70193) - 9 \,(2) \right]$

$\qquad\qquad = 2.5723$

By Adam's corrector method,

$$y_{n+1,c} = y_n + \frac{h}{24} \left[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2} \right]$$

$$\text{(when } n = 3\text{)}, \; y_{4,c} = y_3 + \frac{h}{24} \left[9y'_4 + 19y'_3 - 5y'_2 + y'_1 \right] \qquad (2)$$

$$y'_4 = (x_4)^2 \,(1 + y_4) = (1.4)^2 \,[1 + 2.5871] = 7.0017$$

$\therefore (2) \Rightarrow y_{4,c} = 1.979 + \dfrac{0.1}{24} \left[9\,(7.030716) + 19\,(5.0345) - 5\,(3.60912) + (2.70193) \right]$

$\qquad\qquad = 2.5749$

Example 5.34. **Using Adam's Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ & $y(4.3) = 1.0143$.**

Solution: Here $y' = \dfrac{2 - y^2}{2x}$, $\qquad\qquad\qquad\qquad\qquad \left[\therefore y_{4,p} = 1.0186 \,\&\, y_{4,c} = 1.0187 \right].$

Example 5.35. **Evaluate $y(0.9)$, using Adam Bashforth's predictor-corrector method, given that $\dfrac{dy}{dx} = xy^{\frac{1}{3}}$, $y(1) = 1$, $y(1.1) = 1.106814$, $y(1.2) = 1.22787$ and $y(1.3) = 1.36412$.**

Solution: Here $h = -0.1$, $\qquad\qquad\qquad\qquad \left[\therefore y_{4,p} = 0.906518 \,\&\, y_{4,c} = 0.906520 \right].$

5.11.3 Anna University Questions

1. Given $\dfrac{dy}{dx} = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$, $y(0.2) = 1.2774$, $y(0.3) = 1.5041$. Use Adam's method to estimate $y(0.4)$. (AM10)

   Solution: $\qquad\qquad\qquad\qquad\qquad\qquad \left[y_{4,p} = 1.8341, \; y_{4,c} = 1.8389 \right]$

2. Using Adam's method to find $y(2)$ if $y' = (x + y)/2$, $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$. (MJ13)

   Solution: $\qquad\qquad\qquad\qquad\qquad\qquad \left[y_{4,p} = 6.8708, \; y_{4,c} = 6.8731 \right]$

3. Using Adam's Bashforth method, find $y(4.4)$ given that $5xy' + y^2 = 2$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$. (AM14)

   Solution: $\qquad\qquad\qquad\qquad\qquad\qquad \left[y_{4,p} = 1.0186, \; y_{4,c} = 1.0187 \right]$

5.11.4 Anna University Questions (Taylor's, RK, Adam, Milne)

1. Given $\dfrac{dy}{dx} = xy + y^2$, $y(0) = 1$, $y(0.1) = 1.1169$ and $y(0.2) = 1.2774$, find (i) $y(0.3)$ by Runge-Kutta method of fourth order and (ii) $y(0.4)$ by Milne's method. (ND10)

   Solution: $\qquad\qquad \left[(i)\,y(0.3) = 1.5041, \; (ii)\,y_{4,p} = 1.8344, \; y_{4,c} = 1.8387 \right]$

2. Using Runge-Kutta method of fourth order, find y for $x = 0.1, 0.2, 0.3$ given that $y' = xy + y^2$, $y(0) = 1$. Continue the solution at $x = 0.4$ usign Milne's method. (AM11)

   Solution: $\left[y(0.1) = 1.11689, \; y(0.2) = 1.2774, \; y(0.3) = 1.5041; \; y_{4,p} = 1.8344, \; y_{4,c} = 1.8387 \right]$

3. Solve $y' = x - y^2, y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Taylor's method using the value $h = 0.1$. (AM11)

   Solution: $\left[\text{TS:P}y(0.1) = 0.9138, y(0.2) = 0.8512, y(0.3) = 0.8076; \text{MM}y_{4,p} = 0.7799, y_{4,c} = 0.7797\right]$

4. Given that $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ obtain y for $x = 0.1, 0.2$ and 0.3 by Taylor's series method and find the solution for $y(0.4)$ by Milne's method. (MJ12)

   Solution: $\left[\text{TS: } y(0.1) = 0.995, y(0.2) = 0.9802, y(0.3) = 0.956; \text{MM: } y_{4,p} = 0.9232, y_{4,c} = 0.9232\right]$

5. Using Runge Kutta method of fourth order, find the value of y at $x = 0.2, 0.4, 0.6$ given $\dfrac{dy}{dx} = x^3 + y, y(0) = 2$. Also find the value of y at $x = 0.8$ using Milne's predictor and corrector method. (AM14)

   Solution: $\left[\text{RK: } y(0.2) = 2.073, y(0.4) = 2.452, y(0.6) = 3.023; \text{MM: } y_{4,p} = 4.1664, y_{4,c} = 3.79536\right]$

5.11.5 Assignment problems

1. Using Taylor series method to find $y(0.1)$ if $y' = x^2 + y^2, y(0) = 1$. (MJ13)

2. Using Taylor's series method, find y at $x = 1.1$ by solving the equation $\dfrac{dy}{dx} = x^2 + y^2; y(1) = 2$. Carryout the computations upto fourth order derivative. (AM14)

3. Using Taylor's series method, find y at $x = 0$ if $\dfrac{dy}{dx} = x^2 y - 1, y(0) = 1$. (ND14)

4. Solve $y' = \dfrac{y - x}{y + x}, y(0) = 1$ at $x = 0.1$ by taking $h = 0.02$ by using Euler's method. (MJ13)

5. Using Modified Euler's method, find $y(4.1)$ and $y(4.2)$ if $5x\dfrac{dy}{dx} + y^2 - 2 = 0; y(4) = 1$. (ND12)

6. Apply modified Euler's method to find $y(0.2)$ and $y(0.4)$ given $y' = x^2 + y^2, y(0) = 1$ by taking $h = 0.2$. (ND14)

7. Using Runge-Kutta method of order four, find y when $x = 1.2$ in steps of 0.1 given that $y' = x^2 + y^2$ and $y(1) = 0.5$. (ND13)

8. Consider the second order initial value problem $y'' - 2y' + 2y = e^{2t} \sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$ using Fourth order Runge Kutta algorithm, find $y(0.2)$. (MJ12)

9. Using Runge-Kutta method find $y(0.2)$ if $y'' = xy'^2 - y^2, y(0) = 1, y'(0) = 0, h = 0.2$. (MJ13)

10. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ find the value of $y(0.1)$ by Runge-Kutta's method of fourth order. (ND14)

11. Solve $y(0.1)$ and $z(0.1)$ from the simultaneous differential equations $\dfrac{dy}{dx} = 2y + z; \dfrac{dz}{dx} = y - 3z; y(0) = 0, z(0) = 0.5$ using Runge-Kutta method of the fourth order. (ND12)

12. Use Milne's predictor-corrector formula to find $y(0.4)$, given $\dfrac{dy}{dx} = \dfrac{(1 + x^2)y^2}{2}, y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12$ and $y(0.3) = 1.21$. (AM10)

13. Use Milne's method to find $y(0.8)$, given $y' = \dfrac{1}{x + y}, y(0) = 1$, for $x = 0.1$ and 0.2 correct to four decimal places. (ND13)

14. Given $5xy' + y^2 = 2, y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097, y(4.3) = 1.0142$. Compute $y(4.4)$ using Milne's method. (ND14)

15. Using Adam's method to find $y(2)$ if $y' = (x + y)/2, y(0) = 2, y(0.5) = 2.636, y(1) = 3.595, y(1.5) = 4.968$. (MJ13)

16. Using Adam's Bashforth method, find $y(4.4)$ given that $5xy' + y^2 = 2, y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097$ and $y(4.3) = 1.0143$. (AM14)

17. Given $\dfrac{dy}{dx} = xy + y^2, y(0) = 1, y(0.1) = 1.1169$ and $y(0.2) = 1.2774$, find (i) $y(0.3)$ by Runge-Kutta method of fourth order and (ii) $y(0.4)$ by Milne's method. (ND10)

18. Using Runge Kutta method of fourth order, find the value of y at $x = 0.2, 0.4, 0.6$ given $\dfrac{dy}{dx} = x^3 + y, y(0) = 2$. Also find the value of y at $x = 0.8$ using Milne's predictor and corrector method. (AM14)