AGNI COLLEGE OF TECHNOLOGY

Chennai – 600 130

Department of Mathematics

Third Semester – Common to all Branches

MA6351-TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

ANNA UNIVERSITY IMPORTANT TWO MARKS QUESTIONS AND ANSWERS

UNIT – 1 PART – A QUESTIONS

1. Form the pde by eliminating the arbitrary function from $z^2 - xy = f\left(\frac{x}{z}\right)$

Differentiate the given equation w.r.t. x and y

$$2zp - y = f'\left(\frac{x}{z}\right)\left(\frac{z - xp}{z^2}\right) \quad \ldots\ldots\ldots(1)$$

$$2zq - x = f'\left(\frac{x}{z}\right)\left(\frac{-qx}{z^2}\right) \quad \ldots\ldots\ldots..(2)$$

Dividing (1) by (2), we get

$$\frac{2zp - y}{2zq - x} = \frac{z - xp}{-qx}$$

$$2z^2q = x(qy + z)$$

2. Find the particular integral of $(D^2 - 2DD' + D'^2)z = e^{(x-y)}$

The auxiliary equation is $m^2 - 2m + 1 = 0$ i.e. $m = 1, 1$

$\therefore$ the complementary function is $f_1(y+x) + x f_2(y+x)$

$$\text{Particular Integral} = \frac{1}{D^2 - DD' + D'^2} e^{x-y}$$

$$= \frac{1}{1+2+1} e^{x-y} \qquad \{ \text{put } D = 1,\ D' = -1 \}$$

$\therefore$ the solution is $z = f_1(y+x) + x f_2(y+x) + \frac{1}{4} e^{x-y}$

3. Form the pde by eliminating an arbitrary function from $z = f(x^2+y^2)$

Differentiate the given equation w.r.t. x and y, we get

$p = f'(x^2 + y^2)(2x)$ and $q = f'(x^2 + y^2)(2y)$

Dividing, $\frac{p}{q} = \frac{x}{y}$

4. Solve $\sqrt{p} + \sqrt{q} = 4$

Let $z = ax + by + c$ be the solution where $\sqrt{a} + \sqrt{b} = 4$

$$b = \left(4 - \sqrt{a}\right)^2$$

$\therefore$ $z = ax + \left(4 - \sqrt{a}\right)^2 y + c$ is the complete solution

5. Form the pde from $(x-a)^2 + (y-b)^2 + z^2 = r^2$ by eliminating the arbitrary constants a and b.

Differentiate the given equation w.r.t x and y, we get

$2(x-a) + 2zp = 0 \quad \Rightarrow \quad (x-a) = -zp$

$2(y-b) + 2zq = 0 \quad \Rightarrow \quad (y-b) = -zq$

substituting this in the given equation, we get

$(zp)^2 + (zq)^2 + z^2 = r^2$

6. Form the pde by eliminating the constants a and b from $z = (x^2 + a^2)(y^2+b^2)$

Differentiate the given equation w.r.t x and y, we get

$p = 2x\,(y^2 + b^2) \quad \Rightarrow \quad (y^2 + b^2) = \dfrac{p}{2x}$

$q = 2y\,(x^2 + a^2) \quad \Rightarrow \quad (x^2 + a^2) = \dfrac{q}{2y}$

using this in the given equation, we get $pq = 4xyz$

7. Solve the partial differential equation $pq = x$.

Let $q = a$. Then $p.a = x$

i.e. $p = \dfrac{x}{a}$

But $dz = p\,dx + q\,dy$

$dz = \dfrac{x}{a}\,dx + a\,dy$

Integrating, $z = \dfrac{x^2}{2a} + ay + c$ is the complete solution

8. Form the pde by eliminating the arbitrary function from $z = xy + f(x^2 + y^2)$

Differentiate the given equation w.r.t. x and y, we get

$p = y + f'(x^2+ y^2)(2x)$ and $q = x + f'(x^2+ y^2)(2y)$

$$\frac{p-y}{q-x} = \frac{x}{y}$$

$$py - qx = y^2 - x^2$$

9. Eliminate the arbitrary function from the pde $z = f(x^2 + y^2)$

Differentiate the given equation w.r.t. x and y, we get

$p = f'(x^2+y^2)(2x)$ and $q = f'(x^2+y^2)(2y)$

$$\frac{p}{q} = \frac{x}{y}$$

$$py = qx$$

10. Solve $p + 2q = 4$

Let $z = ax + by + c$ be the solution where $a + 2b = 4$

$a = 4 - 2b$

$\therefore$ $z = (4-2b)x + by + c$ is the complete solution

11. Form the pde of all planes cutting equal intercepts from the x and y axes.

The equation of plane is $\frac{x}{a}+\frac{y}{a}+\frac{z}{b}=1$

Differentiate this w.r.t. x and y, we get

$\frac{1}{a}+\frac{1}{b}=0$ and $\frac{1}{a}+\frac{q}{b}=0$

$\frac{1}{a} = -\frac{1}{b}$ and $\frac{1}{a} = -\frac{q}{b}$

$\therefore$ equating, we get $-\frac{p}{b} = -\frac{q}{b}$

$p = q$

12. Solve $(D^3 - 2D^2 D')z = 0$.

The auxiliary equation is $m^3 - 2m^2 = 0$

$m^2(m-2) = 0$

$m = 0, 0, 2$

$\therefore$ the solution is $z = f_1(y) + x f_2(y) + f_3(y+2x)$

13. By eliminating the arbitrary constants, form the partial differential equation from $z = (x^2 + a)(y^2+b)$

Differentiate the given equation w.r.t x and y, we get

$p = 2x(y^2 + b)$ $\Rightarrow$ $(y^2 + b) = \frac{p}{2x}$

$q = 2y(x^2 + a)$ $\Rightarrow$ $(x^2 + a) = \frac{q}{2y}$

using this in the given equation, we get $pq = 4xyz$

14. Find the particular integral of $(D^2 + 3DD' + 2D'^2)z = (x + y)$

$$\text{Particular Integral} = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{D^2\left[1+\frac{3D}{D'}+\frac{2D'^2}{D^2}\right]}(x+y)$$

$$= \frac{1}{D^2}\left[1+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)\right]^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1-\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)+\ldots\ldots\right](x+y)$$

$$= \frac{1}{D^2}\left[1-\frac{3D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}(x+y)-\frac{3D'}{D^3}(x+y)$$

$$= \frac{yx^2}{2}-\frac{x^3}{3}$$

15. Solve (D – 2D′) (D – 2D′ +1) z = 0.

Comparing this with ($D - m_1 D' - c_1$) ($D - m_2 D' - c_2$) = 0, we get

$c_1 = 0$, $c_2 = -1$, $m_1 = 2$, $m_2 = 2$

∴ the solution is $z = e^{c_1x} f_1(y+m_1x) + e^{c_2x} f_2(y+m_2x)$

$z = e^{0x} f_1(y+2x) + e^{-x} f_2(y+2x)$

16. Form the pde by eliminating the arbitrary constants a and b from $(x-a)^2 + (y-b)^2 = z^2\cot^2\alpha$

Differentiate the given equation w.r.t. x and y

$2(x-a) = 2zp\cot^2\alpha \quad \Rightarrow \quad (x-a) = zp\cot^2\alpha$

$2(y-b) = 2zq\cot^2\alpha \quad \Rightarrow \quad (y-b) = zq\cot^2\alpha$

using this in the given equation, we get

$(zp\cot^2\alpha)^2 + (zq\cot^2\alpha)^2 = z^2\cot^2\alpha$

$p^2 + q^2 = \tan^2\alpha$

17. Form the pde by eliminating f from the relation $z = x + y + f(x^2 + y^2)$

$p = 1 + f'(x^2+y^2)(2x)$ and $q = 1 + f'(x^2+y^2)(2y)$

$$\frac{p-1}{q-1} = \frac{x}{y}$$

$$py - qx = y - x$$

18. Form the pde by eliminating arbitrary functions from z = f(x) g(y)

Differentiate w.r.t x and y twice, we get

$p = f'(x)\,g(y)$ $q = f(x)\,g'(y)$ $s = f'(x)\,g'(y)$

$pq = f'(x)\,g(y)\; f(x)\,g'(y)$

$pq = zs$

19. Solve $(4D^2 - 4DD' + D'^2)z = 0$

The auxiliary equation is $4m^2 - 4m + 1 = 0$ i.e. $m = \frac{1}{2}, \frac{1}{2}$

$\therefore$ the solution is $z = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$

20. Form the pde by eliminating the arbitrary constants from $z = a^2 x + a y^2 + b$

Differentiate the given equation w.r.t x and y, we get

$p = a^2$ …….(1)

$q = 2ay \Rightarrow a = \frac{q}{2y} \Rightarrow a^2 = \frac{q^2}{4y^2}$ ……..(2)

equating (1) and (2) we get $p = \frac{q^2}{4y^2}$

21. Find the complete integral of p – q = 0.

Let the solution is $z = ax + by + c$ where $a - b = 0$
$b = a$

$\therefore z = ax + ay + c$ is the complete solution

22. Form the pde from z = x+ y + f(xy) by eliminating the arbitrary function f.

Differentiate w.r.t x & y, we get

$p = 1 + y f'(xy)$ and $q = 1 + x f'(xy)$

$$\frac{p-1}{q-1} = \frac{y}{x}$$

23. Solve (D – D′) (D +2D′+1)z = 0

Comparing the given pde with (D – m_1 D′ –c_1) (D – m_2 D′ –c_2) = 0, we get
Here $c_1 = 0$, $c_2 = 1$ and $m_1 = -1$, $m_2 = -2$

$\therefore$ the solution is $z = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$

$$z = e^{0x} f_1(y - x) + e^{x} f_2(y - 2x)$$

24. Form the pde by eliminating a and b from z = (x + a)^2 + (y + b)^2.

Differentiate the given equation w.r.t. x and y, we get
p = 2(x+a) and q = 2(y+b)
using this in the given equation, we get

$$z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

25. Solve (D^2 – 2 D D′ + D'^2) z = 0

The auxiliary equation is $m^2 - 2m + 1 = 0$ i.e. m = 1, 1
$\therefore$ the complementary function is $f_1(y+x) + x f_2(y+x)$

26. Form the partial differential equation by eliminating the arbitrary constants from (x – a)^2 + (y – b)^2 + z^2 = 1

Differentiate the given equation w.r.t x and y, we get
2(x–a) + 2zp = 0 $\Rightarrow$ (x–a) = – zp
2(y–b) + 2zq = 0 $\Rightarrow$ (y–b) = – zq
substituting this in the given equation, we get $(zp)^2 + (zq)^2 + z^2 = 1$

27. Solve $\frac{\partial^2 z}{\partial x^2} = xy$

Integrating w.r.t. x twice, we get $\frac{\partial z}{\partial x} = \frac{x^2}{2} y + f(y)$

$$z = \frac{x3}{6} y + x f(y) + g(y)$$

28. Find the complete solution of the pde $p^2 + q^2 - 4pq = 0$.

Let z = ax + by+ c be the solution where $a^2 + b^2 - 4ab = 0$

$$a = \frac{4b \pm \sqrt{16b^2 - 4b^2}}{2}$$

$$a = b(2 \pm \sqrt{3})$$

∴ z = b (2 ± √3) x + by + c is the complete solution

29. Find the general solution of $\frac{\partial^2 z}{\partial x^2} = 0$

Integrating w.r.t. x twice, we get $\frac{\partial z}{\partial x} = f(y)$

$$z = xf(y) + g(y)$$

30. Form the pde by eliminating the arbitrary constants a & b from z = a(x+y)+b

Differentiate the given equation w.r.t. x and y, we get
p = a and q = a
Equating, we get p = q

31. Solve $\frac{\partial^2 z}{\partial x^2} = \sin y$

Integrating w.r.t. x twice, we get $\frac{\partial z}{\partial x} = x\sin y + f(y)$

$$z = \frac{x^2}{2}\sin y + xf(y) + g(y)$$

32. Find the complete integral of $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$

Given equation is z = px + qy + (pq) $^{3/2}$
This is clairaut's equation. ∴ the complete solution is z = ax + by + (ab) $^{3/2}$
Differentiate w.r.t a and b, we get

$0 = x + \frac{3}{2}(ab)^{\frac{1}{2}} b$ and $0 = y + \frac{3}{2}(ab)^{\frac{1}{2}} a$

$x = -\frac{3}{2}(ab)^{\frac{1}{2}} b$ and $y = -\frac{3}{2}(ab)^{\frac{1}{2}} a$

Dividing, $\frac{x}{y} = \frac{b}{a}$ i.e. b = a$\frac{x}{y}$

Multiplying, $xy = \frac{9}{4}(ab)^2$

$$x\,y = \frac{9}{4}a^2\left(\frac{a^2x^2}{y^2}\right)$$

$$a^4 = \frac{4y^3}{9x} \qquad \text{i.e.} \quad a = \left(\frac{4y^3}{9x}\right)^{\frac{1}{4}}$$

$$\therefore \quad b = \left(\frac{4y^3}{9x}\right)^{\frac{1}{4}} \frac{x}{y}$$

$$= \left(\frac{4x^3}{9y}\right)^{\frac{1}{4}}$$

$$z = x\left(\frac{4y^3}{9x}\right)^{\frac{1}{4}} + y\left(\frac{4x^3}{9y}\right)^{\frac{1}{4}} + \left(\frac{4y^3}{9x}\cdot\frac{4x^3}{9y}\right)^{\frac{3}{8}}$$

$$= 4\left(\frac{4xy}{9}\right)^{\frac{3}{4}}$$

33. Obtain the p.d.e. by eliminating f and g from z = f(x + i t) + g (x – i t)

Differentiate the given equation w.r.t. x and t

$p = f'(x+it) + g'(x-it)$ $\qquad$ $q = i f'(x+it) - i g'(x-it)$

$r = f''(x+it) + g''(x-it)$ $\qquad$ $t = -f''(x+it) - g''(x-it)$

$t = -\,[\,f''(x+it) + g''(x-it)\,]$

$t = -r$ $\qquad$ i.e $\quad t + r = 0$ $\qquad$ i.e. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial t^2} = 0$

34. Solve $\quad 4\dfrac{\partial^2 z}{\partial x^2} - 12\dfrac{\partial^2 z}{\partial x\partial y} + 9\dfrac{\partial^2 z}{\partial y^2} = 0$

The auxiliary equation is $4m^2 - 12m + 9 = 0$ $\qquad$ i.e. $\quad m = \dfrac{3}{2}, \dfrac{3}{2}$

$\therefore$ the solution is $z = f_1\left(y + \dfrac{3}{2}x\right) + x f_2\left(y + \dfrac{3}{2}x\right)$

35. Form the p.d.e. of all spheres whose radius is r and centres lie on the z-axis.

The equation of the sphere is $x^2 + y^2 + (z-a)^2 = r^2$

Differentiate w.r.t. x and y, we get

$2x + 2(z-a)p = 0$ and $2y + 2(z-a)q = 0$

$z-a = -\frac{x}{p}$ and $z-a = -\frac{y}{q}$

Equating, we get $-\frac{x}{p} = -\frac{y}{q}$

$qx = py$

36. Find the complete integral of $(1-x)p + (2-y)q = 3-z$

The auxiliary equation is $\frac{dx}{1-x} = \frac{dy}{2-y} = \frac{dz}{3-z}$

Consider $\frac{dx}{1-x} = \frac{dy}{2-y}$

Integrating, $-\log(1-x) = -\log(2-y) + \log C1$

$$\log\left(\frac{2-y}{1-x}\right) = \log C1$$

$$\frac{2-y}{1-x} = C1$$

similarly equating last two, we get $\frac{3-z}{2-y} = C2$

$\therefore$ the solution is $\phi(C1,\ C2) = 0$

37. Find the complete integral of $p - y^2 = q + x^2$

Given $p - x^2 = q + y^2 = k$

i.e. $p - x^2 = k$ and $q + y^2 = k$

i.e. $p = x^2 + k$ and $q = k - y^2$

We know that $z = \int p\,dx + \int q\,dy$

$$z = \int x^2 + k\ dx + \int k - y^2\ dy$$

$$z = kx + \frac{x^3}{3} + ky - \frac{y^3}{3} + c$$

38.Find the pde by eliminating the arbitrary function from $\phi\,[z^2 - xy, x/z] = 0$

Let $u = z^2 - xy$ and $v = \frac{x}{z}$

$\therefore\ \phi(u,v) = 0$ and the pde is given by $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$

$$\begin{vmatrix} 2zp - y & \frac{z - xp}{z^2} \\ 2zq - x & -\frac{xq}{z^2} \end{vmatrix} = 0$$

$$qy + z - p^2 = 0$$

39. Find the singular sol. of $z = px + qy + p^2 - q^2$

This is clairaut's form. $\therefore$ The complete solution is $z = ax + by + a^2 - b^2$

Differentiate this w.r.t a and b, we get

$0 = x + 2a \quad \Rightarrow \quad a = -\frac{x}{2}$

$0 = y - 2b \quad \Rightarrow \quad b = \frac{y}{2}$

$\therefore\ z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$ is the singular solution

40. Find the p.d.e. by eliminating the arbitrary function from z = f(xy).

Differentiate w.r.t. x and y, we get

$p = f'(xy)\,y$ and $q = f'(xy)\,x$

$\frac{p}{y} = f'(xy)$ and $\frac{q}{x} = f'(xy)$

Equating, we get $\frac{p}{y} = \frac{q}{x}$

41. Write down the complete solution of $z = px + qy + c\sqrt{(1 + p^2 + q^2)}$

Consider $z = ax + by + c\sqrt{1+a^2+b^2}$

Differentiate w.r.t a and b, we get

$0 = x + c\left(1+a^2+b^2\right)^{-\frac{1}{2}}a$ and $0 = y + c\left(1+a^2+b^2\right)^{-\frac{1}{2}}b$

$x = -c\left(1+a^2+b^2\right)^{-\frac{1}{2}}a$ and $y = -c\left(1+a^2+b^2\right)^{-\frac{1}{2}}b$

$$\frac{x}{y} = \frac{a}{b} \quad \text{i.e.} \quad b = \frac{ay}{x}$$

$\therefore$ the complete solution is $z = ax + \frac{ay}{x}y + c\sqrt{1+a^2+\left(\frac{ay}{x}\right)^2}$

42. Find the solution of $px^2 + qy^2 = z^2$

This is lagrange's equation. $\therefore$ the auxiliary equation is $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$

Consider $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integrating, $\frac{1}{x} = \frac{1}{y} + C1$ i.e. $\frac{1}{x} - \frac{1}{y} = C1$

Similarly, consider $\frac{dy}{y^2} = \frac{dz}{z^2}$

Integrating, $\frac{1}{y} = \frac{1}{z} + C2$ i.e. $\frac{1}{y} - \frac{1}{z} = C2$

$\therefore$ the solution is $\phi(c1, c2) = 0$

43. Find the p.d.e. of all planes passing through the origin.

The equation of plane passing through the origin is $z = ax + by$

Differentiate w.r.t. x and y, we get $p = a$ and $q = b$

$\therefore$ the pde is $z = px + qy$

44. Find the particular integral of $(D^3 - 3D^2D' - 4DD'^2 + 12D'^3)z = \sin(x+2y)$

Particular Integral = $\dfrac{1}{D^3 - 3D^2D' - 4DD'^2 + 12D'^3}\sin(x+2y)$

$= \dfrac{1}{-D + 3D' + 16D - 48D'}\sin(x+2y)$ { put $D^2 = -1$, $D'^2 = -4$, $DD' = -2$ }

$= \dfrac{1}{15D - 45D'}\sin(x+2y)$

$= \dfrac{15D + 45D'}{225D^2 - 2025D'^2}\sin(x+2y)$

$= \dfrac{15\cos(x+2y) + 90\cos(x+2y)}{225 + 8100}$

45. Form the PDE by eliminating the constants a & b from $z = ax^n + by^n$.

Differentiate the given equation w.r.t. a and b, we get

$p = nax^{n-1}$ and $q = nby^{n-1}$

$a = \dfrac{p}{nx^{n-1}}$ and $b = \dfrac{q}{ny^{n-1}}$

using this in the given equation, we get $z = \dfrac{p}{n}x + \dfrac{q}{n}y$

46. Solve $(D^3 + D^2D' - DD'^2 - D'^3)z = 0$

The auxiliary equation is $m^3 + m^2 - m - 1 = 0$

$(m-1)(m+1)(m+1) = 0$

$m = 1, -1, -1$

$\therefore$ the solution is $z = f_1(y+x) + f_2(y-x) + x f_3(y-x)$

47. Solve $(D^2 - DD' + D' - 1)z = 0$

Given equation can be written as $(D-1)(D-D'+1)z = 0$

Comparing this with $(D - m_1D' - c_1)(D - m_2D' - c_2) = 0$, we get

Here $c_1 = -1$, $c_2 = 1$ and $m_1 = 0$, $m_2 = -1$

$\therefore$ the solution is $z = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$

$z = e^{-x} f_1(y) + e^{x} f_2(y - x)$

48. Eliminate the arbitrary function f from $z = f\left(\dfrac{xy}{z}\right)$ and form the PDE

Differentiate the given equation w.r.t. x and y, we get

$p = f'\left(\frac{xy}{z}\right)\left[\frac{zy - xyp}{z^2}\right]$ and $q = f'\left(\frac{xy}{z}\right)\left[\frac{zx - xyq}{z^2}\right]$

Dividing, we get $\frac{p}{q} = \frac{zy - xyp}{zx - xyq}$

$p\,x = q\,y$

49. Find the complete integral of p + q = p q

Let z = ax + by+ c be the solution where $a + b = ab$

$a = b\,(a - 1)$

$b = \frac{a}{a-1}$

$\therefore$ $z = ax + \frac{a}{a-1}y + c$ is the complete solution

50. Solve $(D^3 - 3DD'^2 + 2D'^3)z = 0$

The auxiliary equation is $m^3 - 3m + 2 = 0$

$(m - 1)(m - 1)(m + 2) = 0$

$m = 1, 1, -2$

$\therefore$ the solution is $z = f_1(y + x) + x\,f_2(y + x) + f_3(y - 2x)$

51. Find the PDE of the spheres having their centers on the line x = y = z

The equation of the sphere is $(x - a)^2 + (y - a)^2 + (z - a)^2 = r^2$

Differentiate w.r.t. x and y, we get

$2(x - a) + 2(z - a)p = 0$ and $2(y - a) + 2(z - a)q = 0$

$(x - a) = -(z - a)p$ and $(y - a) = -(z - a)q$

$x + zp = a(1 + p)$ and $y + zq = a(1 + q)$

$a = \frac{x + zp}{1 + p}$ and $a = \frac{y + zq}{1 + q}$

Equating, we get $\frac{x + zp}{1 + p} = \frac{y + zq}{1 + q}$

52. $(D^3 - 2D^2D' - 4DD'^2 + 8D'^3)z = 0$

The auxiliary equation is $m^3 - 2m^2 - 4m + 8 = 0$

$(m - 2)(m - 2)(m + 2) = 0$

m = 2, 2, –2

∴ the solution is $z = f_1(y + 2x) + x\, f_2(y + 2x) + f_3(y - 2x)$

UNIT – 2 PART A QUESTIONS

1. Write the conditions for a function f(x) to satisfy for the existence of a Fourier series.

i. f(x) is single valued and finite in (c , c+2*l*)
ii. f(x) is continuous or piecewise continuous with finite number of finite discontinuities in (c , c+2*l*)
iii. f(x) has no or finite number of maxima or minima in (c , c+2*l*)

This is known as Drichlet's conditions

2. If $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx,$ **deduce that** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$

Putting x = π, we get

$$\pi^2 = \frac{\pi^2}{3} + 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..\right\}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$$

3. If $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ **is the fourier series of f(x) = x in (–1,1), find** $a_3^2 + b_3^2$.

Since the given f(x) is odd function, $a_3 = 0$.

$$b_3 = \frac{1}{1}\int_{-1}^{1} f(x)\ \sin\ 3\pi x\ dx$$

$$= 2\int_{0}^{1} x\ \sin\ 3\pi x\ dx$$

$$= 2\left\{(x)\left(-\frac{\cos 3\pi x}{3\pi}\right) - (1)\left(-\frac{\sin 3\pi x}{9\pi^2}\right)\right\}_0^1$$

$$= \frac{2}{3\pi}$$

Therefore $a_3^2 + b_3^2 = \frac{4}{9\pi^2}$

4. If f(x) = $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx)$ **is the Fourier cosine series of f(x) in 0 < x < π, state the corresponding Parseval's identity.**

$$\int_0^{\pi}[f(x)]^2\,dx=(\pi-0)\left\{\frac{a_0^2}{4}+\frac{1}{2}\sum_1^{\infty}a_n^2\right\}$$

5. Find b_n in the expansion of $f(x) = x^2$ as a fourier series in $(-\pi, \pi)$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\,dx$$

$= 0$ { since x^2 sinnx is odd }

6. Find the constant term in the expansion of $\cos^2x$ as a fourier series in $(-\pi,\pi)$

$$\cos^2 x = \frac{1}{2}+\frac{\cos 2x}{2}$$

$\therefore$ the constant term is $\frac{1}{2}$

7. Find the root mean square value of $f(x) = x^2$ in $(0, l)$

$$\text{RMS value} = \sqrt{\frac{\int_0^{l}[f(x)]^2\,dx}{l-0}} = \sqrt{\frac{1}{l}\int_0^{l}x^4\,dx} = \sqrt{\frac{\pi^4}{5}}$$

8. Find the Fourier sine series of $f(x) = 1$, $0 < x < \pi$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx$$

$$=\frac{2}{n\pi}\left[1-(-1)^n\right]$$

9. Write the Parseval's identity for the half range sine series of f(x) defined in $(0, l)$

$$\int_0^{l}[f(x)]^2\,dx=(l-0)\left\{\frac{1}{2}\sum_1^{\infty}\left(b_n^2\right)\right\}$$

10. Let f(x) be defined in $(0,2\pi)$ by $f(x)=\begin{cases}\dfrac{1+\cos x}{\pi-x}, & 0<x<\pi\\ \cos x, & \pi<x<2\pi\end{cases}$ **and $f(x+2\pi) = f(x)$.**

Find the value of $f(\pi)$.

Since x = π is a point of discontinuity, the fourier series converges to $\frac{f(\pi-)+f(\pi+)}{2}$

$$f(\pi-) = \underset{x\to\pi-}{Lt}\ \frac{1+\cos x}{\pi - x}$$

$$= \underset{x\to\pi-}{Lt}\ \frac{-\sin x}{-1} \quad \text{by L'Hospital rule}$$

$$= 0$$

$$f(\pi-) = \underset{x\to\pi+}{Lt}\ \cos x$$

$$= -1$$

$$\therefore f(\pi) = -\frac{1}{2}$$

11. State Drichlet's conditions for the convergence of the Fourier series of f(x) in (0,2π) with period 2π.

i. f(x) is bounded function of period 2π
ii. f(x) has finite number of maxima or minima
iii. f(x) has finite number of points of discontinuity
then the Fourier series of f(x) converges to f(x) at all points where f(x) is continuous. Also, it converges to the average value of the right and left hand limits of f(x) at each point where f(x) is discontinuous.

12. State Euler's formula for the fourier constants of a function defined in(c, c+2*l*)

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi}{l}x\, dx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi}{l}x\, dx$$

13. Find the half range cosine series of f(x) = cosx, 0 < x < π

The cosine series is f(x) = cosx { since $a_0 = 0$, $a_n = 0$, n >1 and $a_1 = 1$ }

14. Obtain the first term of the fourier series for the function f(x) = x^2, –π <x< π

The first term is a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$= \frac{2\pi^2}{3}$

15. To which value, the sine series of f(x) = x^2 in (0, 5) converges at x = 5.

Consider F(x) = f(x) in $0 < x < 5$

= – f(–x) in $-5 < x < 0$

F(x) = $-x^2$ in $-5 < x < 0$

= x^2 in $0 < x < 5$

Now x = 5 is an end point.

$\therefore$ the Fourier series converges to $\frac{f(-5)+f(5)}{2} = \frac{25-25}{2} = 0$

16. State the parseval's formula for f(x) defined in (–*l*, *l*).

$$2l\left\{\frac{a_0^2}{4}+\frac{1}{2}\sum_{1}^{\infty}\left(a_n^2+b_n^2\right)\right\} = \int_{-l}^{l}[f(x)]^2\,dx$$

17. If f(x) = sin x, –π < x <π find an.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

= 0 , { since f(x) cosnx is odd function }

18. What is the value of a_n in the fourier series expansion of f(x) = x^3, –π <x< π

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

= 0 , since f(x) cosnx is odd function

19. Find the rms value of of f(x) = x^2 in (0, π)

RMS value $= \sqrt{\frac{\int_0^{\pi}[f(x)]^2\,dx}{\pi-0}} = \sqrt{\frac{1}{\pi}\int_0^{\pi}x^4\,dx} = \sqrt{\frac{\pi^4}{5}}$

20. State the half range cosine series for the function f(x) in the interval $0 \le x \le 2l$

$$f(x) = \frac{a_0}{2}+\sum_{1}^{\infty}a_n\cos\frac{n\pi}{2l}x \quad \text{where} \quad a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{2l}\int_0^{2l} f(x)\ \cos\frac{n\pi}{2l} x dx$$

21. If f(x) = 2x in (0,4), then find the value of a_2 in the Fourier series expansion.

Here $2l = 4$ and hence $l = 2$

$$a_2 = \frac{1}{2}\int_0^4 f(x)\cos\frac{2\pi}{2}x\ dx = \int_0^4 x\ \cos\pi x\ dx = \left\{(x)\left(\frac{\sin\pi x}{\pi}\right)-(1)\left(-\frac{\cos\pi x}{\pi^2}\right)\right\}_0^4 = 0$$

22. The Fourier series expansion of f(x) in (0,2π) is f(x) = $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$. Find the root mean square value of f(x) in the interval (0,2π).

By Parseval's theorem,

$$\int_0^{2\pi}[f(x)]^2\,dx = (2\pi - 0)\left\{\frac{a_0^2}{4}+\frac{1}{2}\left(a_n^2+b_n^2\right)\right\}$$

$$\sqrt{\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx} = \sqrt{\frac{a_0^2}{4}+\frac{1}{2}\sum_1^{\infty}\left(a_n^2+b_n^2\right)}$$

RMS value $= \sqrt{\frac{1}{2}\sum_1^{\infty}\frac{1}{n^2}}$ { given $a_0 = a_n = 0$ and $b_n = \frac{1}{n}$ }

23. Define root mean square value of a function f(x) in (0,2*l*)

RMS value $= \sqrt{\dfrac{\int_0^{2l}[f(x)]^2\,dx}{2l-0}}$

24. Determine b_n in the fourier series expansion of f(x) = ½ (π – x) in (0,2π)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\ dx$$

$$= \frac{1}{2\pi}\left\{(\pi - x)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_0^{2\pi} = \frac{1}{n}$$

25. Define r.m.s. value of f(x) in a < x < b.

RMS value $= \sqrt{\dfrac{\int_a^b [f(x)]^2\,dx}{b-a}}$

26. If f(x) is discontinuous at x = a, what value does its fourier series represent at that point.

The value of the fourier series at a point of discontinuity is the average of left and right limits of the function

27. If $f(x) = \begin{cases} \cos x, & 0 < x < \pi \\ 50, & \pi < x < 2\pi \end{cases}$ **and f(x) = f(x+2π) for all x, find the sum of the fourier series of f(x) at x = π.**

Here x = π is a point of discontinuity. ∴ sum of the series = $\dfrac{f(\pi-)+f(\pi+)}{2}$

$= \dfrac{-1+50}{2} = \dfrac{49}{2}$

28. Find the coefficient b_5 of cos5x in the fourier cosine series of the function f(x) = sin 5x in the interval (0,2π).

The fourier cosine series is f(x) = $\dfrac{b_0}{2} + \sum_1^\infty b_n \cos nx$

$$b_n = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_5 = \frac{2}{2\pi}\int_0^{2\pi} \sin 5x \cos 5x\,dx$$

$= 0$

29. If $\cos^3 x = \dfrac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx)$ **in 0 ≤ x ≤ 2π, find the sum of the series** $\dfrac{a_0^2}{4} + \dfrac{1}{2}\sum_1^\infty (a_n^2 + b_n^2)$

By Parseval's theorem, we have

$$\left\{\frac{a_0^2}{4}+\frac{1}{2}\sum_1^{\infty}\left(a_n^2+b_n^2\right)\right\} = \frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cos^6 x\,dx$$

$$= \frac{4}{2\pi}\int_0^{\pi/2}\cos^6 x dx$$

$$= \frac{5}{16}$$

30. The fourier series of x^2 in (0,2) and that of $(x+2)^2$ in (–2,0) are identical or not. Give reasons.

Here both the curves are periodic in the interval (0,2) and are identical. $\therefore$ their Fourier series are identical.

31. Find a_n in expanding e^{-x} as fourier series in $(-\pi,\pi)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{1+n^2}\left[-\cos nx+n\sin nx\right]\right\}_{-\pi}^{\pi}$$

$$= \frac{(-1)^n}{\pi(1+n^2)}\left[e^{\pi}-e^{-\pi}\right]$$

32. State Parseval's Identity of Fourier series.

$$\frac{1}{2\pi}\int_c^{c+2\pi}[f(x)]^2\,dx = \left\{\frac{a_0^2}{4}+\frac{1}{2}\sum_1^{\infty}\left(a_n^2+b_n^2\right)\right\}$$

33. Find the fourier constant bn for x sin x in $(-\pi,\pi)$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \quad \{\text{ since f(x) sin nx is odd function }\}$$

34. State Parseval's identity for the half range cosine expansion of f(x) in (0, *l*)

$$\frac{1}{l}\int_0^l [f(x)]^2\,dx = \left\{\frac{a_0^2}{4}+\frac{1}{2}\sum_1^{\infty}(a_n^2)\right\}$$

35. If f(x) is discontinuous at x = a, what does its fourier series represent at x=a.

The value of the fourier series at a point of discontinuity is the average of left and right limits of the function

36. Find a_n in the cosine series expansion of f(x) = K in (0, 10)

$$a_n = \frac{2}{10}\int_0^{10} f(x)\cos\frac{n\pi}{10}x\,dx = \frac{k}{5}\int_0^{10}\cos\frac{n\pi}{10}x\,dx = 0$$

37. Find the r.m.s. value of the function f(x) = x in (0, *l*)

$$\text{RMS value} = \sqrt{\frac{\int_0^l [f(x)]^2\,dx}{l-0}} = \sqrt{\frac{1}{l}\int_0^l x^2\,dx} = \sqrt{\frac{l^2}{3}}$$

38. What do you mean by harmonic analysis?

The method of finding fourier series of f(x) which is given by numerical values is called harmonic analysis.

39. In the Fourier series expansion of f(x) = $f(x) = \begin{cases} 1+\frac{2}{\pi}x, & -\pi < x < 0 \\ 1-\frac{2}{\pi}x, & 0 < x < \pi \end{cases}$

find the value of b_n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \quad \{\text{ since } f(x)\sin nx \text{ is odd function }\}$$

40. Does f(x) = tanx possess a fourier expansion?

It has no fourier expansion. Because it is discontinuous at infinity only.

41. Find bn in the expansion of x^2 as a fourier series in (−π , π)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx = 0 \quad \{\text{ since f(x) sin nx is odd function }\}$$

42. If f(x) is an odd function in (– *l*, *l*), what are the values of a_0 & a_n ?

$a_0 = 0$ & $a_n = 0$

43. Find the Fourier sine series for the function f(x) = 1 ; 0 < x < π

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin nx\ dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\left[1-(-1)^n\right]$$

44. If the fourier series for the function $f(x) = \begin{vmatrix} 0, 0 < x < \pi \\ \sin x, \pi < x < 2\pi \end{vmatrix}$ **is**

$f(x) = -\frac{1}{\pi} + \frac{2}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + ...\right] + \frac{1}{2}\sin x$, **duduce that**

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ..... = \frac{\pi - 2}{4}$$

Put x = $\frac{\pi}{2}$ in the series. Since x = $\frac{\pi}{2}$ is a point of discontinuity, fourier series converges to $f\left(\frac{\pi}{2}\right) = 0$

$$0 = -\frac{1}{\pi} + \frac{2}{\pi}\left\{-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + ...\right\} + \frac{1}{2}$$

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - ..... = \frac{\pi - 2}{4}$$

45. Find a_0 , a_n in the fourier series expansion of f(x) = x – x^3 in $(-\pi, \pi)$

Given f(x) is odd function. ∴ $a_0 = 0$ & $a_n = 0$.

46. State Parseval's Identity for f(x) as fourier series in (0 , 2*l*)

$$\frac{1}{2l}\int_0^{2l}[f(x)]^2\,dx = \left\{\frac{a_0^2}{4}+\frac{1}{2}\sum_1^{\infty}\left(a_n^2+b_n^2\right)\right\}$$

47. If the fourier series corresponding to f(x) = x in the interval (0,2π) is $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$, without finding the values of a_0 , a_n , b_n find the value of $\left\{\frac{a_0^2}{2}+\sum_1^{\infty}\left(a_n^2+b_n^2\right)\right\}$

By Parseval's theorem, we have

$$2\pi\left\{\frac{a_0^2}{4}+\frac{1}{2}\sum_1^{\infty}\left(a_n^2+b_n^2\right)\right\} = \int_0^{2\pi}[f(x)]^2\,dx$$

$$\pi\left\{\frac{a_0^2}{2}+\sum_1^{\infty}\left(a_n^2+b_n^2\right)\right\} = \int_0^{2\pi}x^2\,dx$$

$$\left\{\frac{a_0^2}{2}+\sum_1^{\infty}\left(a_n^2+b_n^2\right)\right\} = \frac{8\pi^2}{3}$$

48. If f(x) = x + x ^{2} is expressed as a fourier series in (–2 , 2) to which value this series converges at x = 2.

Since x = 2 is an end point, the Fourier series converges to $\frac{f(2)+f(-2)}{2} = 4$

49. If the Fourier series of the function f(x) = x + x ^{2} in $-\pi < x < \pi$ is $\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right]$, then find the value of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+....$

put x = π in the given Fourier series. Since x = π is an end point, the Fourier series converges to $\frac{f(\pi)+f(-\pi)}{2} = \pi^2$

$$\pi^2 = \frac{\pi^2}{3}+\left\{\frac{4}{1^2}+\frac{4}{2^2}+\frac{4}{3^2}+.......\right\}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+....$$

50. To which value the half range sine series corresponding to $f(x) = x^2$ in the interval $(0, 2)$ converges at $x = 2$.

Consider $F(x) = f(x)$ in $0 < x < 2$

$= -f(-x)$ in $-2 < x < 0$

$F(x) = -x^2$ in $-2 < x < 0$

$= x^2$ in $0 < x < 2$

Now x = 2 is an end point. ∴the fourier series converges to $\frac{f(2)+f(-2)}{2} = 0$

51. **If** $x^2 = \frac{\pi^2}{3} + \sum_1^\infty \frac{4(-1)^n}{n^2}\cos nx$ **in** $(-\pi, \pi)$ **find** $\sum_1^\infty \frac{1}{n^4}$

Given $f(x) = x^2$, $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$ and $b_n = 0$

By Parseval's Identity, we have

$$2\pi\left\{\frac{a_0^2}{4} + \frac{1}{2}\sum_1^\infty\left(a_n^2 + b_n^2\right)\right\} = \int_{-\pi}^{\pi}[f(x)]^2\,dx$$

$$\left\{\frac{a_0^2}{4} + \frac{1}{2}\sum_1^\infty\left(a_n^2\right)\right\} = \frac{2}{2\pi}\int_0^\pi x^4\,dx$$

$$\frac{\pi^4}{9} + 8\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{5}$$

$$\sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$$

52. What is the period of the function cos 3x?

The period is $\frac{2\pi}{3}$

53. If f(x) is an odd function in $(-\pi, \pi)$ then the graph of f(x) is symmetric about - - - - - - - -

The graph is symmetric about origin.

54. Examine whether the function f(x) = $\frac{1}{1-x}$ can be expanded in a Fourier series in any interval including x = 1.

At x = 1, f(x) is discontinuous at infinity. $\therefore$ it cannot be expanded as a fourier series.

55. Without evaluating any integral, write the half range series with sine terms for f(x) = $\sin^3 x$ in $(0,\pi)$

$$f(x) = \sin^3 x$$

$$= \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

UNIT – 3 PART A QUESTIONS

1. Write down the three possible solutions of one dimensional heat equation.

$$u(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)Ce^{\alpha^2\lambda^2 t}$$

$$u(x,t) = \left(A\cos\lambda x + B\sin\lambda x\right)Ce^{-\alpha^2\lambda^2 t}$$

$$u(x,t) = (Ax+B)C$$

2. Give three possible solutions of two dimensional steady state heat flow equation.

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(C\cos\lambda y + D\sin\lambda y\right)$$

$$u(x,y) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$u(x,y) = (Ax+B)(Cy+D)$$

3. Write all possible solutions of the one dimentional wave equation $u_{tt} = a^2 u_{xx}$

$$y(x,t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda at} + De^{-\lambda at}\right)$$

$$y(x,t) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(C\cos\lambda at + D\sin\lambda at\right)$$

$$y(x,t) = (Ax+B)(Cx+D)$$

4. Find steady state solution, if the right and left ends of an insulated rod of length 20 cm is kept at 20° C and 80° C respectively.

The solution of heat equation in steady state is u = ax + b
Here u = 20 when x = 0 and u = 80 when x = 20
Using this, we get b = 20 and a = 3
$\therefore$ the solution is u = 3x + 20

5. **A taut string of length 20 cm fastened at both ends, is disturbed from its position of equilibrium by imparting to each of its points an initial velocity of magnitude kx(20–x) for 0<x<20. Formulate the problem mathematically.**

$y(0,t) = 0$ for all $t \geq 0$ $y(20,t) = 0$ for all $t \geq 0$
$y(x,0) = 0$ for $0 \leq x \leq 20$ $y_t(x,0) = kx(20-x)$ for $0 \leq x \leq 20$

6. **Clasify the following II orde pde** $4\frac{\partial^2 u}{\partial x^2} + 8\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - 6\frac{\partial u}{\partial x} - 8\frac{\partial u}{\partial y} - 16u = 0$

Here $A = 4, B = 8, C = 4$
$\Delta = B^2 - 4AC = 64 - 64 = 0.$
$\therefore$ the pde is parabolic

7. **A tightly stretched string with fixed end points x = 0, x = *l* is initially in a position given by** $y(x,0) = y_0 \sin^3\left(\frac{\pi}{l}x\right)$**. If it is released from rest in this position, write the boundary conditions.**

$y(0, t) = 0$ for all $t \geq 0$ $y(l, t) = 0$ for all $t \geq 0$
$y_t(x, 0) = 0$ for $0 \leq x \leq l$ $y(x,0) = y_0 \sin^3\left(\frac{\pi}{l}x\right)$ for $0 \leq x \leq l$

8. **Write down the one dimensional heat equation.**

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

9. **If the ends of the string of length *l* are fixed and the mid point of the string is drawn aside through a height b and the string is released from rest. Write the boundary and initial conditions.**

$y(0, t) = 0$ for all $t \geq 0$ $y(l, t) = 0$ for all $t \geq 0$ $y_t(x, 0) = 0$ for $0 \leq x \leq l$

$$y(x,0) = \begin{cases} \frac{2b}{l}x, & \left(0, \frac{l}{2}\right) \\ \frac{2b}{l}(l-x), & \left(\frac{l}{2}, l\right) \end{cases}$$

10. **Find the boundary and initial conditions of a rod of length *l* cm with insulated surface is initially at temperature 100°C and at both ends are kept at 0°C.**

$u(0, t) = 0$ for $t \geq 0$ $u(l, t) = 0$ for $t \geq 0$ $u(x, 0) = 100$ for $0 \leq x \leq l$

11. **Classify the partial differential equation** $4u_{xx} = u_t$.

Here A = 4, B = 0, C = 0

$\Delta = B^2 - 4AC = 0$

$\therefore$ the pde is parabolic

12. Classify the pde $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Here A = 1, B = 0, C = 1

$\Delta = B^2 - 4AC = -4$

$\therefore$ the pde is elliptic

13. What is meant by steady state condition in one dimensional heat flow?

A state in which temperature is constant with respect to time is called steady state

14. A taut string of length 50 cm fastened at both ends, is disturbed from its position of equilibrium by imparting to each of its points an initial velocity of magnitude kx for 0<x<50. Formulate the problem mathematically.

$y(0, t) = 0$ for all $t \geq 0$ $\quad$ $y(50, t) = 0$ for all $t \geq 0$

$y(x, 0) = 0$ for $0 \leq x \leq 50$ $\quad$ $y_t(x, 0) = kx$ for $0 \leq x \leq 50$

15. Classify the pde: $(x+1)z_{xx} + \sqrt{2}(x+y+1)z_{xy} + (y+1)z_{yy} + yz_x - xz_y + 2\sin x = 0$.

Here A = x+1, $B = \sqrt{2}(x+y+1)$, C = y+1

$\Delta = B^2 - 4AC$

$= 2(x+y+1)^2 - 4(x+1)(y+1)$

$= 2(x^2 + y^2 - 1)$

For all x and y such that $x^2 + y^2 = 1$, $\Delta = 0$. Then the pde is parabolic

For all x and y such that $x^2 + y^2 < 1$, $\Delta < 0$. Then the pde is elliptic

For all x and y such that $x^2 + y^2 > 1$, $\Delta > 0$. Then the pde is hyperbolic

16. Verify that $y = \cosh \lambda x \cosh(-\lambda at)$ is a solution of $y_{tt} = a^2 y_{xx}$

$y_t = \lambda a \cosh \lambda x \sinh \lambda at$

$y_{tt} = \lambda^2 a^2 \cosh \lambda x \cosh \lambda at$

$y_x = \lambda \sinh \lambda x \cosh \lambda at$

$y_{xx} = \lambda^2 \cosh \lambda x \cosh \lambda at$

Here $y_{tt} = a^2 y_{xx}$. $\therefore$ y is a solution

17. If an insulated rod of length 50 cm has its ends A and B are maintained at 0°C and 100° C respectively, find steady state solution

The solution of heat equation in steady state is $u = ax + b$

Here u = 0 when x = 0 and u = 100 when x = 50

Using this, we get b = 0 and a = 2

$\therefore$ the solution is $u = 2x$

18. Classify the pde: $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - 12\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 7u = x^2 + y$

Here A = 1, B = 4, C = 4

$\Delta = B^2 - 4AC = 16 - 16 = 0$

$\therefore$ the pde is parabolic

19. Classify the pde: $3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - u = 0$

Here A = 3, B = 4, C = 6

$\Delta = B^2 - 4AC = 16 - 72 < 0$

$\therefore$ the pde is elliptic

20. A rod of length 50 cm long with insulated sides has its ends A and B kept at 20° C and 70° C respectively. Find steady state temperature distribution of the rod.

The solution of heat equation in steady state is u = ax + b

Here u = 20 when x = 0 and u = 70 when x = 50

Using this, we get b = 20 and a = 1

$\therefore$ the solution is u = x + 20

21. A taut string of length L cm fastened at both ends, is disturbed from its position of equilibrium by imparting to each of its points an initial velocity of magnitude kx(L–x) for 0<x<L. Formulate the problem mathematically.

$y(0, t) = 0$ for all $t \geq 0$ $\quad$ $y(L, t) = 0$ for all $t \geq 0$

$y(x, 0) = 0$ for $0 \leq x \leq L$ $\quad$ $y_t(x, 0) = kx(L-x)$ for $0 \leq x \leq L$

22. Classify the pde $3u_{xx} + 4u_{xy} + 3u_y - 2u_x = 0$

Here A = 3, B = 4, C = 0

$\Delta = B^2 - 4AC = 16 > 0$

$\therefore$ the pde is hyperbolic

23. The ends A and B of a rod of length 10 cm long have their temperature kept at 20° C and 70° C respectively. Find steady state temperature distribution of the rod.

The solution of heat equation in steady state is u = ax + b

Here u = 20 when x = 0 and u = 70 when x = 10

Using this, we get b = 20 and a = 5

$\therefore$ the solution is u = 5x + 20

24. Write the laplace equation in polar coordinates.

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$

25. Solve the equation 3 u_x + 2 u_y = 0, given that u(x,0) = 4 e^{-x} by the method of separation of variables.

Let u = XY be the solution where X = X(x) and Y = Y(y)

Then $u_x = X'Y$ and $u_y = XY'$

$\therefore \; 3X'Y + 2XY' = 0$

$3X'Y = -2XY'$

$$3\frac{X'}{X} = -2\frac{Y'}{Y} = k$$

$$\frac{X'}{X} = \frac{k}{3} \quad \text{and} \quad \frac{Y'}{Y} = -\frac{k}{2}$$

Integrating, we get

$$\log X = \frac{k}{3}x + \log A \quad \text{and} \quad \log Y = -\frac{k}{2}y + \log B$$

$$X = A\,e^{\frac{k}{3}x} \quad \text{and} \quad Y = B\,e^{-\frac{k}{2}y}$$

$$\therefore \; u = AB\,e^{\frac{k}{3}x}\,e^{-\frac{k}{2}y} \quad \text{--------(1)}$$

But given that u = 4 e^{-x} when y = 0

$$\therefore \; 4e^{-x} = AB\,e^{\frac{k}{3}x} \quad \text{---------(2)}$$

comparing (1) and (2), we get AB = 4 and k = –3

$$\therefore \; u = 4\,e^{-x+\frac{3}{2}y}$$

26. Write the one dimensional wave equation with initial and boundary conditions in which the initial position of the string is f(x) and the initial velocity imparted at each point x is g(x).

The one dimentional wave equation $y_{tt} = a^2 y_{xx}$

$y(0, t) = 0$ for all $t \geq 0$ $\quad$ $y(l, t) = 0$ for all $t \geq 0$

$y(x,0) = f(x)$ for $0 \leq x \leq l$ $\quad$ $y_t(x,0) = g(x)$ for $0 \leq x \leq l$

27. State any two assumptions involved in deriving one dimensional wave equation.

The string is perfectly flexible. It can transmit only tension but not bending or shearing forces.

The effect of friction is negligible. The motion takes place entirely in one plane say xy plane.

28. How many conditions are required to solve $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

Three conditions are required

29. Write down the governing equation of two dimensional steady state heat conduction.

$u_{xx} + u_{yy} = 0$

30. Classify the p.d.e. $u_{xx} = (1/\alpha^2)\, u_t$

Here A = 1, B = 0, C = 0

$\Delta = B^2 - 4AC = 0$

$\therefore$ the pde is parabolic

31. Write the initial conditions of the wave equation if the string has an initial displacement but no initial velocity.

$y(x,0) = f(x)$ for $0 \le x \le l$ $\quad$ $y_t(x,0) = 0$ for $0 \le x \le l$

32. Write the p.d.e. governing one dimensional heat equation.

$u_t = \alpha^2 u_{xx}$

33. State one dimensional heat equation with initial and boundary conditions.

The one dimensional heat eqhation is $u_t = \alpha^2 u_{xx}$

$u(0, t) = 0$ for all $t \ge 0$ $\quad$ $u(l, t) = 0$ for all $t \ge 0$ $\quad$ $u(x, 0) = f(x)$ for $0 \le x \le l$

34. State any two solutions of the laplace equation by the method of variable separable.

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(C\cos\lambda y + D\sin\lambda y\right)$$

$$u(x,y) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$u(x,y) = (Ax+B)(Cy+D)$$

35. In steady state conditions derive the solution of one dimensional heat flow equation.

The one dimensional heat eqhation is $u_t = \alpha^2 u_{xx}$

In steady state the temperature is constant w.r.t time t. $\therefore u_t = 0$

$\therefore$ The heat equation becomes $u_{xx} = 0$. i.e. $\frac{d^2u}{dx^2} = 0$

Integrating w.r.t. x two times, we have u = ax + b

36. A string of length of length 2*l* is fastened at both ends. The mid point of the string is displaced to a distance 'b' and released from rest in this posision. Write the initial conditions.

$y_t(x, 0) = 0$ for $0 \le x \le 2l$

$$y(x,0) = \begin{cases} \frac{b}{l}x, & (0,l) \\ \frac{b}{l}(2l-x), & (l,2l) \end{cases}$$

37. In the one dimentional heat equation $u_t = \alpha^2 u_{xx}$ what does α^2 stand for ?

$\alpha^2 = \dfrac{k}{\rho c}$, diffusivity of the substance

38. What is the basic difference between the solutions of one dimentional wave equation and one dimentional heat equation.

The solution of one dimensional wave equation is periodic in nature but the solution of one dimensional heat equation is not periodic.

39. Write the steady state heat flow equation in two dimension in Cartesian and polar form.

$u_{xx} + u_{yy} = 0$ is in Cartesian form and $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$ in polar form

40. In the diffusion equation $u_t = \alpha^2 u_{xx}$ what does α^2 stand for ?

Refer Q. No.2 of May 2006 – MA 231

41. State fourier law of heat conduction.

The rate at which heat flows through an area is proportional to the area and to the temperature
gradient normal to the area. This constant proportionality is known as the thermal conductivity
(k) of the material

42. In the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ what does c^2 stands for?

$c^2 = \dfrac{Tension}{mass}$

43. State any two laws which are assumed to derive one dimensional heat equation.

Heat flows from a higher temperature to lower temperature

The amount of heat required to produce a given temperature change in a body is proportional to the Mass of the body and the temperature change.

44. Classify : $u_{xx} + x\, u_{yy} = 0$

Here A = 1, B = 0, C = x

$\Delta = B^2 - 4AC = -4x$

When x = 0, Δ = 0 and hence the pde is parabolic

When x < 0, Δ > 0 and hence the pde is hyperbolic

When x > 0, Δ < 0 and hence the pde is elliptic

45. A rod 30 cm long has its ends A and B kept at 20° C and 80° C respectively until steady state conditions prevail. Find the steady state temperature in the rod.

The solution of heat equation in steady state is u = ax + b

Here u = 20 when x = 0 and u = 80 when x = 30

Using this, we get b = 20 and a = 2

∴ the solution is u = 2x + 20

46. Classify the following partial differential equations :

i. $y^2 u_{xx} - 2xy\, u_{xy} + x^2 u_{yy} + 2u_x - 3u = 0$

ii. $y^2 u_{xx} + u_{yy} + (u_x)^2 + (u_y)^2 + 7 = 0$

(i) Here $A = y^2$, $B = -2xy$, $C = x^2$

$\Delta = B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$

∴ the pde is parabolic

(ii) Here $A = y^2$, B = 0, C = 1

$\Delta = B^2 - 4AC = -4y^2$

When y = 0, Δ = 0 and hence the pde is parabolic

When y < 0, Δ < 0 and hence the pde is elliptic

When y > 0, Δ < 0 and hence the pde is elliptic

47. An insulated rod of length 60 cm has its ends at A and B is maintained at 20° C and 80° C respectively. Find the steady state solution of the rod.

The solution of heat equation in steady state is u = ax + b

Here u = 20 when x = 0 and u = 80 when x = 60

Using this, we get b = 20 and a = 1

∴ the solution is u = x + 20

48. Classify the following equations:

i. $4u_{xx} + 4u_{xy} + u_{yy} - 6u_x - 8u_y - 16u = 0$

ii. $u_{xx} + u_{yy} = (u_x)^2 + (u_y)^2$

(i) Here $A = 4$, $B = 4$, $C = 1$

$\Delta = B^2 - 4AC = 16 - 16 = 0$

∴ the pde is parabolic

(ii) Here $A = 1$, $B = 0$, $C = 1$

$\Delta = B^2 - 4AC = -4$

∴ the pde is elliptic

49. Write any two solutions of the laplace equation obtained by the method of separation of variables.

$$u(x,y) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(C\cos\lambda y + D\sin\lambda y\right)$$

$$u(x,y) = \left(A\cos\lambda x + B\sin\lambda x\right)\left(Ce^{\lambda y} + De^{-\lambda y}\right)$$

$$u(x,y) = (Ax+B)(Cy+D)$$

50. Classify the PDE : (i) $u_{xx} = u_{yy}$ (ii) $u_{xy} = u_x u_y + xy$

(i) Here $A = 1$, $B = 0$, $C = -1$

$\Delta = B^2 - 4AC = 4$

∴ the pde is hyperbolic

(ii) Here $A = 0$, $B = 1$, $C = 0$

$\Delta = B^2 - 4AC = 1$

∴ the pde is hyperbolic

UNIT – 4 PART A QUESTIONS

1. Write the Fourier transform pair

$$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$ and

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

2. Find the Fourier sine transform of $f(x) = e^{-ax}$, a>0

$$F_S\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}\sin sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^{\infty}$$

$$=\sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}$$

3. If F(s) is the FT of f(x), then show that F [f(x–a)] = e ^{ias} F(s)

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\ e^{(t+a)is}dt \qquad \{\text{ put } x-a = t \text{ and } dx = dt \}$$

$$= e^{ias}\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\ e^{ist}dt$$

$$= e^{ias}\ F(s)$$

4. State the Fourier sine integral theorem

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\lambda x\int_0^{\infty} f(t)\sin\lambda t\ dt\ d\lambda$$

5. Find the Fourier transform of f(x), defined as $f(x) = \begin{cases} k, |x| < a \\ 0, |x| > a \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} k\ e^{isx}dx$$

$$= \frac{k}{\sqrt{2\pi}}\left\{\frac{e^{isx}}{is}\right\}_{-a}^{a}$$

$$= \frac{k}{s\sqrt{2\pi}}\{2\sin sa\}$$

6. Find the Fourier cosine transform of f(x) = e $^{-ax}$, x>0

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a\cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

7. State the inversion formula for a Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

8. State the convolution theorem of Fourier transform

The Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transforms. F[f(x) * g(x)] = F[f(x)] . F[g(x)]

9. If F{ f(x) } = F(s), then prove that F{ f(x)cosax } = ½ [F(s–a) + F(s+a)]

$$F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \; e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2} \right) e^{isx} dx$$

$$= \frac{1}{2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{(s+a)ix} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{(s-a)ix} dx \right\}$$

= ½ [F(s–a) + F(s+a)]

10. If F{ f(x) } = F(s), then prove that F{ f(x+a) } = e^{-iaa} F(s)

$$F[f(x+a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+a) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; e^{(t-a)is} dt \qquad \{ \text{put } x + a = t \text{ and } dx = dt \}$$

$$= e^{-ias} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \; e^{ist} dt$$

$$= e^{-ias} F(s)$$

11. Find the Fourier transform of $f(x) = \begin{cases} e^{ikx}, a < x < b \\ 0, \ x < a \ and \ x > b \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{i(s+k)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{i(s+k)x}}{i(s+k)} \right\}_a^b$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{i(s+k)b}}{i(s+k)} - \frac{e^{i(s+k)a}}{i(s+k)} \right\}$$

12. Find the Fourier sine transform of $\frac{1}{x}$

$$F_S[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{1}{x} \sin sx \ dx$$

$$= \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2}$$

13. State Parseval's identity in Fourier transforms

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(s)|^2 ds$$

14. Find the Fourier transform of f(x), defined as $f(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1 \ e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{e^{isx}}{is} \right\}_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{e^{isa}-e^{-isa}}{is}\right\}$$

$$= \frac{1}{s\sqrt{2\pi}}\{2\sin sa\}$$

15. State Fourier integral theorem

If f(x) is piecewise continusouly differentiable and absolutely integrable in (–∝ , ∝), then

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\ e^{i(x-t)s}dt\ ds \quad \text{(OR)} \quad f(x)=\frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\ dt\ d\lambda$$

16. If $F_s(s)$ is the Fourier Sine Transform of f(x), show that

F_s\{ f(x) cos ax \} = ½ \{ $F_s(s+a) + F_s(s-a)$ \}

$$F_S[f(x)\cos ax]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \cos ax\ \sin sx dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \left(\sin(s+a)x+\sin(s-a)x\right)\ dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \left(\sin(s+a)x\right)\ dx\ +$$

$$\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \left(\sin(s-a)x\right)\ dx$$

= ½ \{ $F_s(s+a) + F_s(s-a)$ \}

17. If F(s) is the FT of f(x), show that $F\left\{e^{\frac{ib}{a}x} f\left(\frac{x}{a}\right)\right\}=aF(as+b)$, a>0

$$F[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[e^{\frac{ib}{a}x} f\left(\frac{x}{a}\right)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{ib}{a}x} f\left(\frac{x}{a}\right)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ibt} f(t)\ e^{iats}a\ dt$$

$$= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{(as+b)it}\, dt$$

$= a\, F(as+b)$

18. If $F_c(s)$ is the Fourier Cosine Transform of f(x), show that $F_c\{ f(x) \cos ax \} = ½ \{ F_c(s+a) + F_c(s–a) \}$

$$F_C\left[f(x)\cos ax\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \cos ax\ \cos sx dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \left(\cos(s+a)x + \cos(s-a)x\right)\ dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \left(\cos(s+a)x\right)\ dx\ +$$

$$\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \left(\cos(s-a)x\right)\ dx$$

$= ½ \{ F_C(s+a) + F_C(s–a) \}$

19. Write the Fourier transform pair

$$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

20. State the shifting properties on Fourier transform

If F{ f(x) } = F(s), then F{ f(x+a) } = e^{-iaa} F(s)

If F{ f(x) } = F(s), then F{ f(x–a) } = e^{iaa} F(s)

21. Write down the Fourier Cosine transform pair of formulae

$$F_C\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\ \cos sx\ dx \quad \text{and} \quad f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_C(s)\ \cos sx\ ds$$

22. Prove that $F [e^{iax} f(x)] = F(s+a)$, where $F[f(x)] = F(s)$

$$F\left[f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[e^{iax} f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax} f(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$$

$= F(s+a)$

23. Find the Fourier cosine transform of f(x) = x for 0 < x < 1
= 2 – x for 1 < x < 2
= 0 for x > 2

$$F_C[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos sx dx + \sqrt{\frac{2}{\pi}} \int_1^2 (2-x) \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ (x)\left(\frac{\sin sx}{s}\right) - (1)\left(-\frac{\cos sx}{s^2}\right) \right\}_0^1 + \sqrt{\frac{2}{\pi}} \left\{ (2-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(-\frac{\cos sx}{s^2}\right) \right\}_1^2$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2\cos s - \cos 2s - 1}{s^2} \right]$$

24. Find the Fourier sine transform of f(x) = 1 in (0 , *l*)

$$F_S[f(x)] = \int_0^l f(x) \sin \frac{s\pi}{l} x \, dx$$

$$= \int_0^l 1 \sin \frac{s\pi}{l} x \, dx$$

$$= \left\{ -\frac{\cos \frac{s\pi}{l} x}{\frac{s\pi}{l}} \right\}_0^l$$

$$= \left\{ -\frac{l}{s\pi} \cos \frac{s\pi}{l} x \right\}_0^l$$

$$= \frac{l}{s\pi} [1 - \cos s\pi]$$

25. Prove that F [f(ax)] = $\frac{1}{a} F\left(\frac{s}{a}\right)$, a>0

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) \, e^{isx} dx$$

$$= \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\, dt \qquad \{ \text{put } ax = t \text{ and } dx = \frac{1}{a} dt \}$$

$$= \frac{1}{a} F\left(\frac{s}{a}\right)$$

26. State the Fourier transform of the derivatives of a function.

$$F\left[\frac{d}{dx} f(x)\right] = -is\, F[f(x)]$$

27. Find the Fourier transform of $e^{-a|x|}$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, (\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, (\cos sx)\, dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\, (i\sin sx)\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ax}\, (\cos sx)\, dx$$

$$= \frac{2}{\sqrt{2\pi}}\left\{\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right\}_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}$$

28. Find (a). F { x^n f(x) } (b). $F\left[\frac{d^n}{dx^n} f(x)\right]$ **in terms of F.T. of f(x)**

(a) $$F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\frac{d}{ds} F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, \frac{\partial}{\partial s} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, (ix) e^{isx}\, dx$$

$$\frac{d^2}{ds^2} F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, (ix)^2\, e^{isx}\, dx$$

$$\frac{d^n}{ds^n}F[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\ (ix)^n\ e^{isx}dx$$

$$= (i)^n \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\ (x^n)e^{isx}dx$$

$$= (i)^n\ F[x^n f(x)]$$

$$F[x^n f(x)] = \frac{1}{(i)^n}\frac{d^n}{ds^n}F[f(x)]$$

(b) $$F\left[\frac{d}{dx}f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\ d[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left(e^{isx} f(x)\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\ (is)\ e^{isx}dx\right\}$$

$$= -\frac{is}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} f(x)\ e^{isx}dx\right\}$$

similarly,

$$F\left[\frac{d^2}{dx^2}f(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f''(x)\ e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{isx}\ d[f'(x)]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left(e^{isx} f'(x)\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\ (is)\ e^{isx}dx\right\}$$

$$= -\frac{is}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty} e^{isx}\ d[f(x)]\right\}$$

$$= -\frac{is}{\sqrt{2\pi}}\left(-\frac{is}{\sqrt{2\pi}}\right)\left\{\int_{-\infty}^{\infty} f(x)\ e^{isx}dx\right\}$$

$$F\left[\frac{d^2}{dx^2}f(x)\right] = \left(-\frac{is}{\sqrt{2\pi}}\right)^2\left\{\int_{-\infty}^{\infty} f(x)\ e^{isx}dx\right\}$$

$$\therefore\ F\left[\frac{d^n}{dx^n}f(x)\right] = \left(-\frac{is}{\sqrt{2\pi}}\right)^n\left\{\int_{-\infty}^{\infty} f(x)\ e^{isx}dx\right\}$$

29. Solve the integral equation $\int^{\infty}$ f(x) cos λx dx = $e^{-\lambda}$

Multiply the given equation by $\sqrt{\frac{2}{\pi}}$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\ \cos\lambda x\ dx = \sqrt{\frac{2}{\pi}}\ e^{-\lambda}$$

$$F_C[\ f(x)\ ] = \sqrt{\frac{2}{\pi}}\ e^{-\lambda}$$

$$f(x) = F_C^{-1}\left\{\sqrt{\frac{2}{\pi}} e^{-\lambda}\right\}$$

$$= \frac{2}{\pi}\int_0^\infty e^{-\lambda}\cos\lambda x\ d\lambda$$

$$= \frac{2}{\pi}\left\{\frac{e^{-\lambda}}{1+x^2}\left(-\cos\lambda x + x\sin\lambda x\right)\right\}_0^\infty$$

$$= \frac{2}{\pi}\frac{1}{1+x^2}$$

30. **If $F_C(s)$ is the Fourier cosine transform of f(x), prove that the Fourier cosine transform of f(a x) is $\frac{1}{a}F_C\left(\frac{s}{a}\right)$**

$$F_C\left[f(x)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx dx$$

$$F_C\left[f(ax)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(ax)\cos sx dx$$

$$= \frac{1}{a}\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\ \cos\left(\frac{s}{a}\right)t\ dt$$

$$= \frac{1}{a}F_C\left(\frac{s}{a}\right)$$

UNIT – 5 PART A QUESTIONS

1. **Define the unit step sequence. Write its Z transform.**

The unit step function is defined as $u(n) = \begin{cases} 0, & t<0 \\ 1, & t\ge 0\end{cases}$

$$\mathbf{Z}\,[\,u(n)\,] = \sum_{n=0}^{\infty} u(n)\left(\frac{1}{z}\right)^n$$

$$= 1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+.....$$

$$= \left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{z}{z-1}$$

2. Form a difference equation by eliminating the constant A from $y_n = A.3^n$

Given $y_n = A.3^n$

$y_{n+1} = A\,3\,.\,3^n$

$= 3\,y_n$

$y_{n+1} - 3\,y_n = 0$

3. State the convolution theorem for Z transforms

If W(n) is the convolution of two sequences x(n) and y(n), then

$\mathbf{Z}\,[\,W(n)\,] = \mathbf{Z}\,[\,x(n)\,]\,.\,\mathbf{Z}\,[\,y(n)\,]$

4. Find the Z transform of the sequence na^n

We know that $\mathbf{Z}\,[\,n\,] = \dfrac{z}{(z-1)^2}$

$$\mathbf{Z}\,[\,n\,a^n\,] = \left\{\,\mathbf{Z}[n]\,\right\}_{z\to\frac{z}{a}}$$

$$= \left\{\frac{z}{(z-1)^2}\right\}_{z\to\frac{z}{a}}$$

$$= \frac{az}{(z-a)^2}$$

5. Find the Z transform of $\sin\dfrac{n\pi}{2}$

We know that $\mathbf{Z}[\,\sin n\theta\,] = \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$

$$\mathbf{Z}\left[\,\sin n\frac{\pi}{2}\,\right] = \frac{z}{z^2+1}$$

6. Find the difference equation generated by $y_n = a + b.2^n$

Given $y_n = a + b.2^n$

$y_{n+1} = a + 2\,b.2^n$

$y_{n+2} = a + 4\,b.2^n$

Eliminating a and b, we get $\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 1 & 2 \\ y_{n+2} & 1 & 4 \end{vmatrix} = 0$

$$y_{n+2} - 3y_{n+1} + 2y_n = 0$$

7. Find the Z transform of $x(n) = \begin{cases} \dfrac{a^n}{n!} & for\ n \geq 0 \\ 0 & otherwise \end{cases}$

$$Z[x(n)] = Z\left[\frac{a^n}{n!}\right]$$

$$= \sum_{n=0}^{\infty} \left(\frac{a^n}{n!}\right)\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{\left(az^{-1}\right)^n}{n!}$$

$$= 1 + \frac{\left(az^{-1}\right)}{1!} + \frac{\left(az^{-1}\right)^2}{2!} + \ldots\ldots$$

$$= e^{\frac{a}{z}}$$

8. If $F(z) = \dfrac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$**, find f(0)**

By Initial value theorem, we know that $f(0) = \underset{z\to\infty}{Lt}\, F(z)$

$$f(0) = \underset{z\to\infty}{Lt} \frac{1}{z\left(1-\frac{1}{2z}\right)\left(1-\frac{1}{4z}\right)\left(1-\frac{3}{4z}\right)}$$

$$= 0$$

9. Find the Z transform of 3^n

$$Z[3^n] = \sum_{n=0}^{\infty} 3^n \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^n$$

$$= 1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+.....$$

$$= \left(1-\frac{3}{z}\right)^{-1}$$

$$= \frac{z}{z-3}$$

10. Does the Z transform of n! exist? Justify your answer.

Z transform of a sequence f(n) exist if $\left|\frac{f(n+1)}{f(n)}\right| < |z|$ as $n \to \infty$

$$\left|\frac{f(n+1)}{f(n)}\right| = \left|\frac{(n+1)!}{n!}\right|$$

$= n+1 \to \infty$ as $n \to \infty$

$\therefore$ Z transform of n! does notexist

11. Find Z [f(n)], where f(n) = n for n = 0, 1, 2, …..

$$Z\,[\,f(n)\,] = \sum_{n=0}^{\infty} f(n)\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^n$$

$$= \left(\frac{1}{z}\right)+\left(\frac{2}{z^2}\right)^2+\left(\frac{3}{z^3}\right)+.....$$

$$= \frac{1}{z}\left\{1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+....\right\}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-2}$$

$$= \frac{z}{(z-1)^2}$$

12. State the initial value theorem in Z transforms.

Initial Value Theorem in Z transform is $f(0) = \underset{z\to\infty}{Lt} F(z)$ where Z [f(n)] = F(z)

13. Define Z transform of the sequence { f(n) }

The Z transform of the sequence { f(n) } , n = 0, 1, 2, …. is defined as

$Z\,[\,f(n)\,] = \sum_{n=0}^{\infty} f(n)\left(\frac{1}{z}\right)^n$ = F(z), where z is an arbitrary complex number

14. Form a difference equation by eliminating arbitrary constants from $u_n = a\,2^{n+1}$

Given $u_n = a\,2^{n+1}$

$u_{n+1} = a\,2\,.\,2^{n+1}$

$u_{n+1} = 2\,u_n$

$u_{n+1} - 2\,u_n = 0$

15. Express $Z\,[\,f(n+1)\,]$ in terms of $\overline{f}(z)$

$Z\,[\,f(n+1)\,] = z\,\{\,F(z) - f(0)\,\}$

16. Find $Z\left[\frac{1}{n}\right]$

$$Z\left[\frac{1}{n}\right] = \sum_{n=0}^{\infty}\frac{1}{n}\left(\frac{1}{z}\right)^n$$

$$= \left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\ldots$$

$$= -\log\left(1 - \frac{1}{z}\right)$$

$$= \log\frac{z}{z-1}$$

17. Find the inverse Z transform of $\dfrac{z}{(z^2 + 7z + 10)}$

Given $F(z) = \dfrac{z}{(z+2)(z+5)}$

$$\text{Res}\,\{\,z^{n-1}F(z)\,;\,z = -2\,\} = \underset{z\to -2}{Lt}\,(z+2)\frac{z^n}{(z+2)(z+5)} = \frac{(-2)^n}{3}$$

$$\text{Res}\,\{\,z^{n-1}F(z)\,;\,z = -5\,\} = \underset{z\to -5}{Lt}\,(z+5)\frac{z^n}{(z+2)(z+5)} = \frac{(-5)^n}{-3}$$

$$f(n) = \text{sum of residues} = \frac{(-2)^n}{3} - \frac{(-5)^n}{3}$$

18. Find the Z transform of (n + 2)

$Z\,[\,n+2\,] = Z\,[\,n\,] + Z\,[\,2\,]$

$$= \frac{z}{(z-1)^2} + 2\frac{z}{z-1}$$

19. State the final value theorem in Z transform

The final value theorem in Z transform is $\underset{n\to\infty}{Lt}\, f(n) = \underset{z\to 1}{Lt}\, (z-1)F(z)$

20. State and prove initial value theorem in Z-transform

By definition of Z transform

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)\left(\frac{1}{z}\right)^n$$

$$F(z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots\ldots$$

Taking limit $z \to \infty$ on both sides, we get

$$\underset{z\to\infty}{Lt}\, F(z) = f(0)$$

21. Find the Z-transform of (n+1)(n+2)

$$Z[(n+1)(n+2)] = Z[n^2 + 3n + 2]$$

$$= Z[n^2] + Z[n] + Z[2]$$

$$= \frac{z^2+z}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{z-1}$$

22. Find Z [e^{-iat}]

We know that $Z[e^{-iat} f(t)] = Z[f(t)]_{z\to ze^{aT}}$

$$Z[e^{-iat}] = Z[1]_{z\to ze^{iaT}}$$

$$= \frac{z}{z-1}\bigg|_{z\to ze^{iaT}}$$

$$= \frac{z\,e^{iaT}}{ze^{iaT} - 1}$$

23. Show that Z [a^n f(n)] = $F\left(\frac{z}{a}\right)$

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)\left(\frac{1}{z}\right)^n = F(z)$$

$$Z[a^n f(n)] = \sum_{n=0}^{\infty} a^n f(n)\left(\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} f(n) \left(\frac{a}{z}\right)^n$$

$$= \sum_{n=0}^{\infty} f(n) \left(\frac{1}{\frac{z}{a}}\right)^n$$

$$= F\left(\frac{z}{a}\right)$$

24. Show that Z [n f(n)] = $-z \dfrac{d}{dz}$ F(z) where F(z) = Z [f(n)]

We know that $F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$\frac{d}{dz}F(z) = -\sum_{n=0}^{\infty} n\, f(n)\, z^{-n-1}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty} nf(n)\, z^{-n}$$

$$-z\frac{d}{dz}F(z) = \text{Z}[\,n\, f(n)\,]$$

25. If Z [f(n)] = F(z), then show that Z [f(–n)] = $F\left(\frac{1}{z}\right)$

We know that, by definition, $\text{Z}\,[\,f(-n)\,] = \sum_{n=-\infty}^{\infty} f(-n)\, z^{-n}$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{m} \quad \{\text{ put } -n = m \}$$

$$= \sum_{m=-\infty}^{\infty} f(m) \left(\frac{1}{z}\right)^m$$

$$= F\left(\frac{1}{z}\right)$$

26. Find the Z transform of n^2

$$\mathbf{Z}[n^2] = \mathbf{Z}[n.n]$$

$$= -z\frac{d}{dz}\,\mathbf{Z}[n]$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= \frac{z^2+z}{(z-1)^3}$$